KRUSKAL-SZEKERES COORDINATES AND THE EVENT HORIZON

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 15; Box 15.4.

The Painlevé-Gullstrand (PG, also known as the global rain system) metric eliminates some of the problems in the Schwarzschild metric (S metric), but the PG metric is not diagonal and as a result, can be a bit difficult to work with. Another metric developed by Martin Kruskal and George Szekeres, and thus known as the Kruskal-Szekeres (KS) metric avoids this problem, although at the cost of introducing coordinates whose meanings aren't exactly intuitive. However, the KS system has a number of other benefits which often make qualitative interpretation of happenings around the event horizon easier.

KS coordinates retain the usual angular coordinates θ and ϕ, but the r and t coordinates of the S metric are replaced with new coordinates u and v which can be defined as:

$$u = \sqrt{\frac{r}{2GM}-1}e^{r/4GM}\cosh\frac{t}{4GM} \tag{1}$$

$$v = \sqrt{\frac{r}{2GM}-1}e^{r/4GM}\sinh\frac{t}{4GM} \tag{2}$$

for $r>2GM$ and

$$u = \sqrt{1-\frac{r}{2GM}}e^{r/4GM}\sinh\frac{t}{4GM} \tag{3}$$

$$v = \sqrt{1-\frac{r}{2GM}}e^{r/4GM}\cosh\frac{t}{4GM} \tag{4}$$

for $r<2GM$. (As I said, they're non-intuitive!) These equations can be inverted to give

$$u^2 - v^2 = \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \tag{5}$$

$$2GM \ln\left|\frac{u+v}{u-v}\right| = t \tag{6}$$

Because the equation for r is transcendental, it's not possible to solve explicitly for r in terms of u and v using simple functions, so r is retained in the expression for the metric, but it should be regarded as a function $r(u,v)$ rather than as an independent coordinate.

From here, we calculate the differentials of these two equations and then substitute for dr and dt in the S metric to get the KS metric. For the r equation:

$$\begin{aligned} 2u\,du - 2v\,dv &= e^{r/2GM}\left(\frac{1}{2GM} + \frac{r}{(2GM)^2} - \frac{1}{2GM}\right) dr && (7)\\ &= e^{r/2GM}\frac{r}{4(GM)^2} dr && (8)\\ dr &= \frac{8(GM)^2}{r} e^{-r/2GM}(u\,du - v\,dv) && (9) \end{aligned}$$

For the t equation, there are two possible cases, due to the absolute value: the argument of the absolute value can be positive or negative. If it's positive, then

$$\begin{aligned} dt &= 2GM\frac{u-v}{u+v}\left[\frac{(du+dv)(u-v)-(u+v)(du-dv)}{(u-v)^2}\right] && (10)\\ &= 2GM\left[\frac{du+dv}{u+v} - \frac{du-dv}{u-v}\right] && (11)\\ &= \frac{2GM}{u^2-v^2}\left[(du+dv)(u-v)-(du-dv)(u+v)\right] && (12)\\ &= \frac{2GM}{r/2GM-1}e^{-r/2GM}\left[2u\,dv - 2v\,du\right] && (13)\\ &= \frac{8(GM)^2}{r-2GM}e^{-r/2GM}\left[u\,dv - v\,du\right] && (14) \end{aligned}$$

where in the fourth line, we used the r equation above to get rid of $u^2 - v^2$.

If the argument is negative, then we are finding the differential of $\ln\left[\frac{v+u}{v-u}\right]$, that is, we have just interchanged u and v. By looking at the second line, we can see this gives the same result for dt. Now it's just a matter of inserting these expressions for dr and dt into the S metric, which is, for radial motion:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 \tag{15}$$

We get

$$dr^2 = \frac{64\,(GM)^4}{r^2} e^{-r/GM} \left(u^2 du^2 + v^2 dv^2 - 2uv\,du\,dv\right) \tag{16}$$

$$\left(1 - \frac{2GM}{r}\right)^{-1} dr^2 = \frac{r}{r - 2GM} \frac{64\,(GM)^4}{r^2} e^{-r/GM} \left(u^2 du^2 + v^2 dv^2 - 2uv\,du\,dv\right) \tag{17}$$

$$= \frac{64\,(GM)^4}{r\,(r - 2GM)} e^{-r/GM} \left(u^2 du^2 + v^2 dv^2 - 2uv\,du\,dv\right) \tag{18}$$

$$dt^2 = \frac{64\,(GM)^4}{(r - 2GM)^2} e^{-r/GM} \left(u^2 dv^2 + v^2 du^2 - 2uv\,du\,dv\right) \tag{19}$$

$$-\left(1 - \frac{2GM}{r}\right) dt^2 = -\frac{r - 2GM}{r} \frac{64\,(GM)^4}{(r - 2GM)^2} e^{-r/GM} \left(u^2 dv^2 + v^2 du^2 - 2uv\,du\,dv\right) \tag{20}$$

$$= -\frac{64\,(GM)^4}{r\,(r - 2GM)} e^{-r/GM} \left(u^2 dv^2 + v^2 du^2 - 2uv\,du\,dv\right) \tag{21}$$

Putting this together, we get

$$\begin{aligned}
ds^2 &= \frac{64\,(GM)^4}{r\,(r - 2GM)} e^{-r/GM} \left[\left(u^2 - v^2\right)\left(du^2 - dv^2\right)\right] && (22)\\
&= \frac{64\,(GM)^4}{r\,(r - 2GM)} e^{-r/GM} \left(\frac{r}{2GM} - 1\right) e^{r/2GM} \left(du^2 - dv^2\right) && (23)\\
&= \frac{32\,(GM)^3}{r} e^{-r/2GM} \left(du^2 - dv^2\right) && (24)
\end{aligned}$$

This metric allows a few things to be seen fairly easily. For example, a photon world line, with $ds^2 = 0$, must have $du = \pm dv$. That is, on a plot of v versus u, photon world lines always have slope ± 1, just like they do in special relativity space-time diagrams. From the r equation above, if r is a constant r_0, then

$$u^2 - v^2 = \left(\frac{r_0}{2GM} - 1\right) e^{r_0/2GM} = const \tag{25}$$

This is the equation of a hyperbola which crosses the u axis and opens to the right if $r_0 > 2GM$ or a hyperbola crossing the v axis and opening upwards if $r_0 < 2GM$. In particular, $r_0 = 0$ gives the hyperbola $u^2 - v^2 = -1$.

Finally, if t is a constant t_0, then

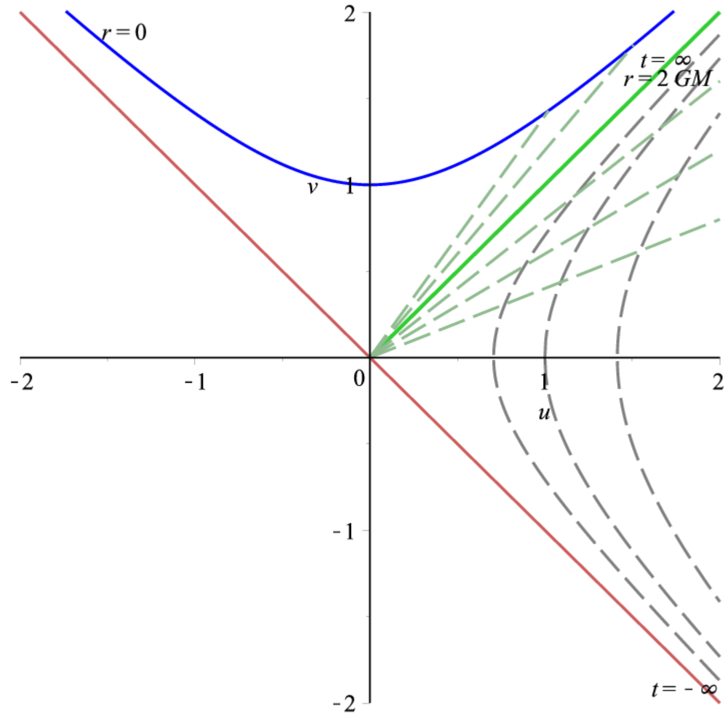
$$2GM \ln\left|\frac{u+v}{u-v}\right| = t_0 \tag{26}$$

from which we get

$$\begin{aligned} \frac{u+v}{u-v} &= const \equiv a && (27) \\ u+v &= au - av && (28) \\ v &= \frac{a-1}{a+1}u && (29) \end{aligned}$$

Thus curves of constant t are straight lines through the origin.

These features can be illustrated on a KS diagram, as shown:



The $r = 0$ hyperbola is shown in blue at the top of the diagram. A few hyperbolas for $r_0 > 2GM$ are shown as grey dashed curves on the right, and a few lines of constant t_0 are the green dashed lines meeting at the origin. The red diagonal line with slope -1 represents $t = -\infty$ and the heavy green line with slope $+1$ is $t = +\infty$. This green line and the upper half of the red line also represent $r = 2GM$ (the event horizon), since the hyperbola reduces to $u^2 - v^2 = 0$ in this case. The region between the $r = 2GM$ lines and the $r = 0$ hyperbola represents space inside the event horizon, but notice that neither the u nor v coordinate does anything pathological at the event horizon boundary, so it is clear that it's easy to cross from outside to inside the event horizon. All world lines move upwards in the diagram, and must end when they reach the $r = 0$ hyperbola.

All photon world lines must be parallel to either the red or green diagonal line. This means that any massive particle's world line must always have a slope whose magnitude is greater than 1 in the diagram, just as in special relativity. In particular, once a particle crosses into the area inside the event horizon, it cannot get out again, since that would require some part of its world line to have a slope less than 1.

It might seem that the condition that any massive particle's world line has a slope greater than 1 would mean that any such particle *must* eventually cross the event horizon, but that's not quite true. For example, if r

is a constant, the world line is one of the grey hyperbolas in the diagram. Such a hyperbola always has a slope greater than 1 (in the upper right quadrant) and yet it never crosses the event horizon. If an object wishes to move away from the black hole, it can manage this by maintaining a world line whose slope is slightly less than the slope of the constant-r hyperbola at each point, yet is still a slope greater than 1. This will continually shift the object to larger values of r. Since all the constant-r hyperbolas have the same asymptote, any desired distance from the black hole can be reached if we carry on this process long enough, since any two constant-r hyperbolas will get infinitesimally close eventually. Because the KS diagram squashes together the curves of r and t far from the black hole, it's not the best diagram for analyzing motion far away; its main strength is in letting us see what happens near and inside the event horizon.

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