BLACK HOLE RADIATION: ENERGY OF EMITTED PARTICLES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 16; Box 16.1.

In Schwarzschild (S) space-time, the energy of a particle is given by

$$e=\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau} \tag{1}$$

Outside the event horizon $r>2GM$ and since the S coordinate t represents time, it must constantly increase as the proper time τ increases, so $dt/d\tau>0$, with the result that $e>0$. Inside the event horizon $r<2GM$, but time and space swap round so $dt/d\tau$ can be either positive or negative, since t is now a space coordinate. The the energy can be positive or negative inside the event horizon. For a single particle moving along a geodesic, e is a constant of the motion, so since $1-\frac{2GM}{r}$ becomes zero at the event horizon, $dt/d\tau$ becomes infinite which results from the fact that the S time becomes infinite at the horizon. After crossing the horizon, $1-\frac{2GM}{r}<0$ so $dt/d\tau$ must also be negative to keep e constant. However, it is possible for the particle to interact with another particle inside the horizon, which could cause $dt/d\tau$ to change sign, resulting in a negative energy.

The idea behind black hole radiation is actually quite simple. Quantum field theory predicts that vacuum fluctuations can occur, in which a particle-antiparticle pair spontaneously appears, essentially out of nothing, provided that the energies of the two particles are equal and opposite; that is, one particle has positive energy E while the other has negative energy $-E$. This phenomenon can occur only for a very short time interval of the order of $\Delta t\sim\hbar/E$ (at this stage, you can just accept all this as god-given, since we haven't studied quantum field theory yet), after which the particle recombines with its antiparticle.

Now suppose this pair creation event occurs very close to the event horizon, and the negative energy particle crosses the horizon before it has a chance to recombine, and that the positive energy particle therefore escapes to infinity. The energy of the black hole is thus reduced by E and the energy E is radiated away to infinity. To see how this works in a simplified model, suppose the pair creation event occurs at some small distance ϵ above the event horizon. We need ϵ to be small enough that the negative energy particle can cross the horizon before it recombines, so we need to estimate how long it takes the particle to travel to the horizon. Since we're outside the horizon, the S coordinate r is still spatial, so we can use the following expression to estimate the proper time required:

$$\frac{dr}{d\tau} = -\sqrt{e^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right)} \tag{2}$$

We've used the minus sign to indicate that r is decreasing. If the particle falls radially, the angular momentum is $\ell = 0$, so the proper time required is

$$\Delta\tau = -\int_{2GM+\epsilon}^{2GM} \frac{dr}{\sqrt{e^2 - \left(1 - \frac{2GM}{r}\right)}} \tag{3}$$

To go further, we need to know e. It's unlikely that the pair is created at rest, but suppose we're in a locally flat reference frame that is released from rest with its origin at $r = 2GM + \epsilon$ and that the pair creation event occurs at this location. An object in the observer's frame has an energy per unit mass of

$$e = \sqrt{1 - \frac{2GM}{2GM + \epsilon}} \tag{4}$$

so for such an object the integral becomes

$$\begin{aligned}
\Delta\tau &= -\int_{2GM+\epsilon}^{2GM} \frac{dr}{\sqrt{1 - \frac{2GM}{2GM+\epsilon} - \left(1 - \frac{2GM}{r}\right)}} && (5) \\
&= -\int_{2GM+\epsilon}^{2GM} \frac{dr}{\sqrt{\frac{2GM}{r} - \frac{2GM}{2GM+\epsilon}}} && (6)
\end{aligned}$$

This integral *can* be done using Maple although the result is a bit too complex to write out here, and then a series expansion of the result around $\epsilon = 0$ gives the leading terms

$$\Delta\tau = 2\sqrt{2GM\epsilon} + \frac{5}{6}\sqrt{\frac{2}{GM}}\epsilon^{3/2} + \mathcal{O}\left(\epsilon^{5/2}\right) \tag{7}$$

If we want just the first term, we can transform the integral a bit using $\rho \equiv r - 2GM$:

$$\Delta\tau = \int_0^{\epsilon} \frac{d\rho}{\sqrt{\frac{2GM}{\rho+2GM} - \frac{2GM}{2GM+\epsilon}}} \tag{8}$$

If we expand each term in the square root in a series and keep terms up to first order in ρ and ϵ we get

$$\frac{2GM}{\rho+2GM} - \frac{2GM}{2GM+\epsilon} = \frac{1}{2GM}(\epsilon-\rho) + \mathcal{O}\left(\epsilon^2\right) + \mathcal{O}\left(\rho^2\right) \tag{9}$$

so to this order, we have

$$\Delta\tau = \int_0^{\epsilon} \frac{d\rho}{\sqrt{(\epsilon-\rho)/2GM}} \tag{10}$$

We can use another substitution $u \equiv \epsilon - \rho$ to get

$$\Delta\tau = \int_0^{\epsilon} \frac{du}{\sqrt{u/2GM}} \tag{11}$$

$$= 2\sqrt{2GM\epsilon} \tag{12}$$

This is the time required for an object released from rest at $r = 2GM + \epsilon$ to reach the event horizon. Presumably if the particle is not at rest when it is created, but is heading towards the event horizon, it will require less time than this to reach it.

Plugging this into the field theory estimate above, we can get an estimate of the energy of the radiated particle:

$$\Delta\tau = \frac{\hbar}{E} \tag{13}$$

$$E = \frac{\hbar}{2\sqrt{2GM\epsilon}} \tag{14}$$

I have to admit I'm not particularly comfortable with this calculation, since $\Delta\tau$ is calculated for an object released at rest from $r = 2GM + \epsilon$ whereas the pair of particles would probably not be produced at rest. However, as an order of magnitude estimate, I suppose it's reasonable.

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