BLACK HOLE HEAT ENGINE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 16; Problem P16.4.

According to thermodynamics, a heat engine is a device that absorbs heat energy Q_H from a reservoir at temperature T_H, uses some of this energy to do work W, and then expels the remaining energy to a colder reservoir at temperature T_C. The efficiency ϵ of the engine is defined as $\epsilon \equiv W/Q_H$, that is, the ratio of the work done to the total energy absorbed in the first place. The laws of thermodynamics state that the efficiency is limited by the condition

$$\epsilon \leq 1 - \frac{T_C}{T_H} \tag{1}$$

A proposal for a perfectly efficient engine is as follows. We collect some radiation inside a perfectly reflecting box from some hot object at temperature T_H that is very far from a black hole. We take the box to the black hole and lower it so that it is just above the event horizon, where we hold it at rest. In this case, the angular momentum of the box is $\ell = 0$ and since it is at rest, $dr/d\tau = 0$, so from the equation

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\frac{l^2}{r^2} - GM\left(\frac{1}{r} + \frac{l^2}{r^3}\right) = \frac{1}{2}\left(e^2 - 1\right) \tag{2}$$

we get

$$-\frac{GM}{r} = \frac{1}{2}\left(e^2 - 1\right) \tag{3}$$

If the box is held right at the event horizon, $r = 2GM$ and thus $e = 0$. If the energy of the radiation when it was added to the box is taken to be an addition to the box's mass, then the energy per unit mass at the starting point is $e = 1$. If we release the radiation from the box into the black hole at the event horizon, we are releasing zero energy, so in some sense we could say that all the original energy was used as work on the trip to the black hole. (Incidentally, although e was originally defined as a constant of particle motion in Schwarzschild (S) space, that applied only for motion on a geodesic, and the current situation doesn't satisfy this, since the box is pulled to a stop before it crosses the horizon. In such a case, e will change with the motion.) The efficiency of the engine could thus be defined as

$$\epsilon = 1 - \frac{e_f}{e_0} \tag{4}$$

where e_0 is the energy per unit mass absorbed initially and e_f is the energy released into the black hole at the end. Since we've seen that we can define a non-zero temperature for a black hole, this appears to violate the laws of thermodynamics. (As a side note, it's not entirely obvious how this process converts the radiation's energy to work. The comparison with a heat engine seems to be from the fact that we are absorbing some radiation at one temperature and then depositing it into the black hole at a different temperature.)

However, this derivation assumes that the box can be brought right up to the event horizon, which isn't really true. For a blackbody, Wien's displacement law says that the peak wavelength emitted by a body at temperature T_H is:

$$\lambda_{max} = \frac{2\pi c\hbar}{4.965 k_B T_H} \tag{5}$$

Since we're interested in order of magnitude calculations in GR units we take $c = 1$ and ignore the numerical constants to get

$$\lambda_{max} \sim \frac{\hbar}{k_B T_H} \tag{6}$$

If the box is a cube, λ_{max} is the minimum size of an edge if the box is to contain radiation of this wavelength, and since no part of the box can be lowered below the event horizon, the closest the centre of the box can approach the black hole is $\lambda_{max}/2$. Using the formula we derived earlier for the physical distance between a point outside the event horizon and the horizon itself, we can find the r coordinate of the box's centre at closest approach.

$$\Delta s = r\sqrt{1 - \frac{2GM}{r}} + 2GM \tanh^{-1}\sqrt{1 - \frac{2GM}{r}} \tag{7}$$

Since we expect r to be close to $2GM$, the argument of the square root terms will be small, so we can use the approximation $\tanh^{-1} x \approx x$ for small x. We therefore get

$$\Delta s = \frac{\lambda_{max}}{2} \approx 4GM\sqrt{1-\frac{2GM}{r}} \tag{8}$$

$$r = 2GM\left(1-\frac{\lambda_{max}^2}{64\,(GM)^2}\right)^{-1} \tag{9}$$

The approximation of r being close to $2GM$ is therefore valid provided $\lambda_{max} \ll 8GM$.

We can rewrite 3 as

$$e^2 = 1-\frac{2GM}{r} \tag{10}$$

$$= \frac{\lambda_{max}^2}{(8GM)^2} \tag{11}$$

Thus if we stop lowering box when its centre is $\lambda_{max}/2$ away from the horizon, the energy is not zero, so we will be releasing some energy into the black hole when we release the radiation. The efficiency is therefore

$$\epsilon = 1-e \tag{12}$$

$$= 1-\frac{\lambda_{max}}{8GM} \tag{13}$$

$$\approx 1-\frac{\hbar}{8k_BT_HGM} \tag{14}$$

This is consistent with the laws of thermodynamics if the black hole's temperature is

$$T_C = \frac{\hbar}{8k_BGM} \tag{15}$$

The actual temperature of a black hole is

$$T = \frac{\hbar}{8\pi k_BGM} \tag{16}$$

so for an order of magnitude estimate, this calculation isn't too bad.