

CHRISTOFFEL SYMBOLS FOR SCHWARZSCHILD METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.6, Problem P17.2.

We've seen that the Christoffel symbols in terms of the metric are given by

$$\Gamma_{ij}^m = \frac{1}{2}g^{ml}(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}) \quad (1)$$

This expression can be cumbersome to work with, since it involves calculating the inverse metric tensor g^{ml} and doing a lot of sums to find each Christoffel symbol. Often an easier way is to exploit the relation between the Christoffel symbols and the geodesic equation.

The geodesic equation is (where a dot above a symbol means the derivative with respect to τ):

$$g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j\dot{x}^i = 0 \quad (2)$$

The following equation is formally equivalent to this:

$$\ddot{x}^m + \Gamma_{ij}^m\dot{x}^j\dot{x}^i = 0 \quad (3)$$

The method for calculating the Christoffel symbols is to work out the terms in 2, divide through by g_{aj} and then compare the result term by term with 3. By doing this we are able to read off the Γ_{ij}^m as the coefficients of $\dot{x}^j\dot{x}^i$ in 2.

Example. We can use this technique to work out the Γ_{ij}^m for the Schwarzschild (S) metric, which is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \quad (4)$$

First, take $a = \phi$ in 2. Since the S metric doesn't depend on ϕ $\partial_\phi g_{ij} = 0$ for all elements. Further, since the S metric is diagonal, $g_{\phi j}$ is restricted to $g_{\phi\phi}$, so the equation becomes

$$g_{\phi\phi}\ddot{\phi} + \partial_i g_{\phi\phi}\dot{\phi}\dot{x}^i = 0 \quad (5)$$

Since $g_{\phi\phi} = r^2 \sin^2 \theta$ there are 2 non-zero derivatives, so this equation expands to

$$r^2 \sin^2 \theta \ddot{\phi} + 2r \sin^2 \theta \dot{\phi} \dot{r} + 2r^2 \sin \theta \cos \theta \dot{\phi} \dot{\theta} = 0 \quad (6)$$

$$\ddot{\phi} + \frac{2}{r} \dot{\phi} \dot{r} + 2 \cot \theta \dot{\phi} \dot{\theta} = 0 \quad (7)$$

By comparing this with 3 we can read off the symbols:

$$\Gamma_{r\phi}^{\phi} + \Gamma_{\phi r}^{\phi} = \frac{2}{r} \quad (8)$$

$$\Gamma_{r\phi}^{\phi} = \Gamma_{\phi r}^{\phi} = \frac{1}{r} \quad (9)$$

$$\Gamma_{\theta\phi}^{\phi} + \Gamma_{\phi\theta}^{\phi} = 2 \cot \theta \quad (10)$$

$$\Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta \quad (11)$$

Here, we've used the symmetry of the Christoffel symbols. Because no other terms appear in the equation, all the other Γ_{ij}^{ϕ} are zero, so the complete set is, where the rows are labelled t , r , θ and ϕ from top to bottom, and the columns the same order from left to right:

$$\Gamma_{ij}^{\phi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot \theta \\ 0 & \frac{1}{r} & \cot \theta & 0 \end{bmatrix} \quad (12)$$

Now consider $a = \theta$. This time, one of the g_{ij} does depend on θ so we will get a contribution from the $\partial_{\theta} g_{\phi\phi}$ term. We get

$$r^2 \ddot{\theta} + 2r \dot{r} \dot{\theta} - \frac{1}{2} r^2 (2 \sin \theta \cos \theta) \dot{\phi}^2 = 0 \quad (13)$$

$$\ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \sin \theta \cos \theta \dot{\phi}^2 = 0 \quad (14)$$

Again, we can read off the symbols to get

$$\Gamma_{ij}^{\theta} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{bmatrix} \quad (15)$$

For $a = r$ things get a bit messier since all four g_{ij} terms depend on r . We get

$$0 = \left(1 - \frac{2GM}{r}\right)^{-1} \ddot{r} - \frac{2GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-2} \dot{r}^2 - \quad (16)$$

$$\frac{1}{2} \left[-\frac{2GM}{r^2} \dot{t}^2 - \frac{2GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-2} \dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2 \right]$$

$$0 = \ddot{r} + \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right) \dot{t}^2 - \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} \dot{r}^2 - \quad (17)$$

$$r \left(1 - \frac{2GM}{r}\right) \dot{\theta}^2 - r\sin^2\theta \left(1 - \frac{2GM}{r}\right) \dot{\phi}^2$$

Comparing terms, we get

$$\Gamma^r_{ij} = \begin{bmatrix} \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r \left(1 - \frac{2GM}{r}\right) & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \left(1 - \frac{2GM}{r}\right) \end{bmatrix} \quad (18)$$

Finally, for $a = t$ the metric is again independent of t so the situation is a lot simpler:

$$-\left(1 - \frac{2GM}{r}\right) \ddot{t} - \frac{2GM}{r^2} \dot{t} \dot{t} = 0 \quad (19)$$

$$\ddot{t} + \frac{2GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} \dot{t} \dot{t} = 0 \quad (20)$$

The symbols are

$$\Gamma^t_{ij} = \begin{bmatrix} 0 & \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (21)$$

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