COVARIANT DERIVATIVE OF A GENERAL TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.2.

We've seen how to define the absolute gradient or **covariant derivative** of a contravariant vector, giving the formula:

$$\boxed{\nabla_j A^k \equiv \frac{\partial A^k}{\partial x^j} + A^i \Gamma_{ij}^k} \tag{1}$$

The covariant derivative of this vector is a tensor, unlike the ordinary derivative. Here we see how to generalize this to get the absolute gradient of tensors of any rank.

First, let's find the covariant derivative of a covariant vector B_i. The starting is to consider $\nabla_j \left(A^i B_i\right)$. The quantity $A^i B_i$ is a scalar, and to proceed we *require* two conditions:

(1) The covariant derivative of a scalar is the same as the ordinary derivative.
(2) The covariant derivative obeys the product rule.

These two conditions aren't derived; they are just required as part of the definition of the covariant derivative.

Using rule 2, we have

$$\nabla_j \left(A^i B_i\right) = \left(\nabla_j A^i\right) B_i + A^i \left(\nabla_j B_i\right) \tag{2}$$

$$= \left(\partial_j A^i + A^k \Gamma_{kj}^i\right) B_i + A^i \left(\nabla_j B_i\right) \tag{3}$$

We now apply rule 1 to the LHS:

$$\nabla_j \left(A^i B_i\right) = \partial_j \left(A^i B_i\right) \tag{4}$$

$$= \left(\partial_j A^i\right) B_i + A^i \left(\partial_j B_i\right) \tag{5}$$

Equating 3 and 5 we get

$$\left(\partial_j A^i + A^k \Gamma^i{}_{kj}\right) B_i + A^i \left(\nabla_j B_i\right) = \left(\partial_j A^i\right) B_i + A^i \left(\partial_j B_i\right) \tag{6}$$

$$B_i A^k \Gamma^i{}_{kj} + A^i \left(\nabla_j B_i\right) = A^i \left(\partial_j B_i\right) \tag{7}$$

$$B_i A^k \Gamma^i{}_{kj} + A^k \left(\nabla_j B_k\right) = A^k \left(\partial_j B_k\right) \tag{8}$$

$$\left[B_i \Gamma^i{}_{kj} + \left(\nabla_j B_k\right)\right] A^k = A^k \left(\partial_j B_k\right) \tag{9}$$

In 8 we've relabelled the dummy index i to k in the second and third terms so we could factor out A^k in the last line. Since the vector A^k is arbitrary, the factors multiplying it on each side must be equal, so we get

$$B_i \Gamma^i{}_{kj} + \left(\nabla_j B_k\right) = \partial_j B_k \tag{10}$$

from which we can get the covariant derivative of B_k:

$$\boxed{\nabla_j B_k = \partial_j B_k - B_i \Gamma^i{}_{kj}} \tag{11}$$

To extend this argument to a tensor of higher rank with mixed indices, we generalize this argument. First, we contract all the indices of the tensor with covariant or contravariant vectors, as appropriate, and then apply the product rule. So for example

$$\nabla_l \left(C^i_{jk} A_i B^j D^k\right) = \nabla_l \left(C^i_{jk}\right) A_i B^j D^k + C^i_{jk} \nabla_l \left(A_i\right) B^j D^k + C^i_{jk} A_i \nabla_l \left(B^j\right) D^k + C^i_{jk} A_i B^j \nabla_l \left(D^k\right) \tag{12}$$

$$= \partial_l \left(C^i_{jk}\right) A_i B^j D^k + C^i_{jk} \partial_l \left(A_i\right) B^j D^k + C^i_{jk} A_i \partial_l \left(B^j\right) D^k + C^i_{jk} A_i B^j \partial_l \left(D^k\right) \tag{13}$$

We now substitute for the covariant derivatives of the vectors in 12 and set the result equal to 13, and cancel terms. We then use the fact that A_i, B^j and D^k are all arbitrary so the factors on each side of the equation multiplying them must be equal. The result is

$$\nabla_l C^i_{jk} = \partial_l C^i_{jk} + \Gamma^i_{lm} C^m_{jk} - \Gamma^m_{lj} C^i_{mk} - \Gamma^m_{lk} C^i_{jm} \tag{14}$$

In general, the rule is that for each contravariant (upper) index in the tensor, there is a positive term with a Christoffel symbol, and for each covariant (lower) index, there is a negative term.

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