CHRISTOFFEL SYMBOLS FOR SCHWARZSCHILD METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.6, Problem P17.2.

We've seen that the Christoffel symbols in terms of the metric are given by

$$\Gamma_{ij}^{m}=\frac{1}{2}g^{ml}\left(\partial_{j}g_{il}+\partial_{i}g_{lj}-\partial_{l}g_{ji}\right) \tag{0.1}$$

This expression can be cumbersome to work with, since it involves calculating the inverse metric tensor g^{ml} and doing a lot of sums to find each Christoffel symbol. Often an easier way is to exploit the relation between the Christoffel symbols and the geodesic equation.

The geodesic equation is (where a dot above a symbol means the derivative with respect to τ):

$$g_{aj}\ddot{x}^{j}+\left(\partial_{i}g_{aj}-\frac{1}{2}\partial_{a}g_{ij}\right)\dot{x}^{j}\dot{x}^{i}=0 \tag{0.2}$$

The following equation is formally equivalent to this:

$$\ddot{x}^{m}+\Gamma_{ij}^{m}\dot{x}^{j}\dot{x}^{i}=0 \tag{0.3}$$

The method for calculating the Christoffel symbols is to work out the terms in 0.2, divide through by g_{aj} and then compare the result term by term with 0.3. By doing this we are able to read off the Γ_{ij}^{m} as the coefficients of $\dot{x}^{j}\dot{x}^{i}$ in 0.2.

Example. We can use this technique to work out the Γ_{ij}^{m} for the Schwarzschild (S) metric, which is

(0.4)

$$ds^{2}=-\left(1-\frac{2GM}{r}\right)dt^{2}+\left(1-\frac{2GM}{r}\right)^{-1}dr^{2}+r^{2}d\theta^{2}+r^{2}\sin^{2}\theta d\phi^{2}$$

First, take $a=\phi$ in 0.2. Since the S metric doesn't depend on ϕ $\partial_{\phi}g_{ij}=0$ for all elements. Further, since the S metric is diagonal, $g_{\phi j}$ is restricted to $g_{\phi\phi}$, so the equation becomes

$$g_{\phi\phi}\ddot{\phi}+\partial_i g_{\phi\phi}\dot{\phi}\dot{x}^i=0 \tag{0.5}$$

Since $g_{\phi\phi}=r^2\sin^2\theta$ there are 2 non-zero derivatives, so this equation expands to

$$\begin{aligned} r^2\sin^2\theta\ddot{\phi}+2r\sin^2\theta\dot{\phi}\dot{r}+2r^2\sin\theta\cos\theta\dot{\phi}\dot{\theta} &= 0 \qquad (0.6)\\ \ddot{\phi}+\frac{2}{r}\dot{\phi}\dot{r}+2\cot\theta\dot{\phi}\dot{\theta} &= 0 \qquad (0.7)\end{aligned}$$

By comparing this with 0.3 we can read off the symbols:

$$\begin{aligned} \Gamma^{\phi}_{\ r\phi}+\Gamma^{\phi}_{\ \phi r} &= \frac{2}{r} \qquad (0.8)\\ \Gamma^{\phi}_{\ r\phi}=\Gamma^{\phi}_{\ \phi r} &= \frac{1}{r} \qquad (0.9)\\ \Gamma^{\phi}_{\ \theta\phi}+\Gamma^{\phi}_{\ \phi\theta} &= 2\cot\theta \qquad (0.10)\\ \Gamma^{\phi}_{\ \theta\phi}=\Gamma^{\phi}_{\ \phi\theta} &= \cot\theta \qquad (0.11)\end{aligned}$$

Here, we've used the symmetry of the Christoffel symbols. Because no other terms appear in the equation, all the other $\Gamma^{\phi}_{\ ij}$ are zero, so the complete set is, where the rows are labelled t, r, θ and ϕ from top to bottom, and the columns the same order from left to right:

$$\Gamma^{\phi}_{\ ij}=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{r}\\ 0 & 0 & 0 & \cot\theta\\ 0 & \frac{1}{r} & \cot\theta & 0\end{bmatrix} \tag{0.12}$$

Now consider $a=\theta$. This time, one of the g_{ij} does depend on θ so we will get a contribution from the $\partial_\theta g_{\phi\phi}$ term. We get

$$\begin{aligned} r^2\ddot{\theta}+2r\dot{r}\dot{\theta}-\frac{1}{2}r^2\left(2\sin\theta\cos\theta\right)\dot{\phi}^2 &= 0 \qquad (0.13)\\ \ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\sin\theta\cos\theta\dot{\phi}^2 &= 0 \qquad (0.14)\end{aligned}$$

Again, we can read off the symbols to get

$$\Gamma_{ij}^{\theta}=\left[\begin{array}{cccc} 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{r} & 0\\ 0 & \frac{1}{r} & 0 & 0\\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{array}\right] \tag{0.15}$$

For $a=r$ things get a bit messier since all four g_{ij} terms depend on r. We get

$$\begin{aligned} 0= & \left(1-\frac{2GM}{r}\right)^{-1}\ddot{r}-\frac{2GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-2}\dot{r}^{2}-\\ & \frac{1}{2}\left[-\frac{2GM}{r^{2}}\dot{t}^{2}-\frac{2GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-2}\dot{r}^{2}+2r\dot{\theta}^{2}+2r\sin^{2}\theta\dot{\phi}^{2}\right] \end{aligned} \tag{0.16}$$

$$\begin{aligned} 0= & \ddot{r}+\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)\dot{t}^{2}-\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1}\dot{r}^{2}-\\ & r\left(1-\frac{2GM}{r}\right)\dot{\theta}^{2}-r\sin^{2}\theta\left(1-\frac{2GM}{r}\right)\dot{\phi}^{2} \end{aligned} \tag{0.17}$$

Comparing terms, we get

$$\Gamma_{ij}^{r}=\left[\begin{array}{cccc} \frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right) & 0 & 0 & 0\\ 0 & -\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ 0 & 0 & -r\left(1-\frac{2GM}{r}\right) & 0\\ 0 & 0 & 0 & -r\sin^{2}\theta\left(1-\frac{2GM}{r}\right) \end{array}\right] \tag{0.18}$$

Finally, for $a=t$ the metric is again independent of t so the situation is a lot simpler:

$$-\left(1-\frac{2GM}{r}\right)\ddot{t}-\frac{2GM}{r^{2}}\dot{r}\dot{t} = 0 \tag{0.19}$$

$$\ddot{t}+\frac{2GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1}\dot{r}\dot{t} = 0 \tag{0.20}$$

The symbols are

$$\Gamma_{ij}^{t}=\left[\begin{array}{cccc} 0 & \frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ \frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{array}\right] \tag{0.21}$$

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