CHRISTOFFEL SYMBOLS FOR SCHWARZSCHILD METRIC

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.6, Problem P17.2.

We've seen that the Christoffel symbols in terms of the metric are given by

$$\Gamma_{ij}^{m}=\frac{1}{2}g^{ml}\left(\partial_{j}g_{il}+\partial_{i}g_{lj}-\partial_{l}g_{ji}\right) \tag{1}$$

This expression can be cumbersome to work with, since it involves calculating the inverse metric tensor g^{ml} and doing a lot of sums to find each Christoffel symbol. Often an easier way is to exploit the relation between the Christoffel symbols and the geodesic equation.

The geodesic equation is (where a dot above a symbol means the derivative with respect to τ):

$$g_{aj}\ddot{x}^{j}+\left(\partial_{i}g_{aj}-\frac{1}{2}\partial_{a}g_{ij}\right)\dot{x}^{j}\dot{x}^{i}=0 \tag{2}$$

The following equation is formally equivalent to this:

$$\ddot{x}^{m}+\Gamma_{ij}^{m}\dot{x}^{j}\dot{x}^{i}=0 \tag{3}$$

The method for calculating the Christoffel symbols is to work out the terms in 2, divide through by g_{aj} and then compare the result term by term with 3. By doing this we are able to read off the Γ_{ij}^{m} as the coefficients of $\dot{x}^{j}\dot{x}^{i}$ in 2.

Example. We can use this technique to work out the Γ_{ij}^{m} for the Schwarzschild (S) metric, which is

$$ds^{2}=-\left(1-\frac{2GM}{r}\right)dt^{2}+\left(1-\frac{2GM}{r}\right)^{-1}dr^{2}+r^{2}d\theta^{2}+r^{2}\sin^{2}\theta d\phi^{2} \tag{4}$$

First, take $a=\phi$ in 2. Since the S metric doesn't depend on ϕ $\partial_{\phi}g_{ij}=0$ for all elements. Further, since the S metric is diagonal, $g_{\phi j}$ is restricted to $g_{\phi\phi}$, so the equation becomes

$$g_{\phi\phi}\ddot{\phi}+\partial_i g_{\phi\phi}\dot{\phi}\dot{x}^i=0 \tag{5}$$

Since $g_{\phi\phi}=r^2\sin^2\theta$ there are 2 non-zero derivatives, so this equation expands to

$$\begin{aligned} r^2\sin^2\theta\ddot{\phi}+2r\sin^2\theta\dot{\phi}\dot{r}+2r^2\sin\theta\cos\theta\dot{\phi}\dot{\theta} &= 0 \quad (6)\\ \ddot{\phi}+\frac{2}{r}\dot{\phi}\dot{r}+2\cot\theta\dot{\phi}\dot{\theta} &= 0 \quad (7)\end{aligned}$$

By comparing this with 3 we can read off the symbols:

$$\begin{aligned} \Gamma^{\phi}_{\ r\phi}+\Gamma^{\phi}_{\ \phi r} &= \frac{2}{r} \quad (8)\\ \Gamma^{\phi}_{\ r\phi}=\Gamma^{\phi}_{\ \phi r} &= \frac{1}{r} \quad (9)\\ \Gamma^{\phi}_{\ \theta\phi}+\Gamma^{\phi}_{\ \phi\theta} &= 2\cot\theta \quad (10)\\ \Gamma^{\phi}_{\ \theta\phi}=\Gamma^{\phi}_{\ \phi\theta} &= \cot\theta \quad (11)\end{aligned}$$

Here, we've used the symmetry of the Christoffel symbols. Because no other terms appear in the equation, all the other $\Gamma^{\phi}_{\ ij}$ are zero, so the complete set is, where the rows are labelled t, r, θ and ϕ from top to bottom, and the columns the same order from left to right:

$$\Gamma^{\phi}_{\ ij}=\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{r}\\ 0 & 0 & 0 & \cot\theta\\ 0 & \frac{1}{r} & \cot\theta & 0\end{bmatrix} \tag{12}$$

Now consider $a=\theta$. This time, one of the g_{ij} does depend on θ so we will get a contribution from the $\partial_\theta g_{\phi\phi}$ term. We get

$$\begin{aligned} r^2\ddot{\theta}+2r\dot{r}\dot{\theta}-\frac{1}{2}r^2\left(2\sin\theta\cos\theta\right)\dot{\phi}^2 &= 0 \quad (13)\\ \ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\sin\theta\cos\theta\dot{\phi}^2 &= 0 \quad (14)\end{aligned}$$

Again, we can read off the symbols to get

$$\Gamma^{\theta}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{15}$$

For $a = r$ things get a bit messier since all four g_{ij} terms depend on r. We get

$$\begin{aligned} 0 = {} & \left(1 - \frac{2GM}{r}\right)^{-1} \ddot{r} - \frac{2GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-2} \dot{r}^2 - \\ & \frac{1}{2}\left[-\frac{2GM}{r^2}\dot{t}^2 - \frac{2GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-2} \dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2\right] \end{aligned} \tag{16}$$

$$\begin{aligned} 0 = {} & \ddot{r} + \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)\dot{t}^2 - \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1} \dot{r}^2 - \\ & r\left(1 - \frac{2GM}{r}\right)\dot{\theta}^2 - r\sin^2\theta\left(1 - \frac{2GM}{r}\right)\dot{\phi}^2 \end{aligned} \tag{17}$$

Comparing terms, we get

$$\Gamma^{r}_{ij} = \begin{bmatrix} \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r\left(1 - \frac{2GM}{r}\right) & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\left(1 - \frac{2GM}{r}\right) \end{bmatrix} \tag{18}$$

Finally, for $a = t$ the metric is again independent of t so the situation is a lot simpler:

$$-\left(1 - \frac{2GM}{r}\right)\ddot{t} - \frac{2GM}{r^2}\dot{r}\dot{t} = 0 \tag{19}$$

$$\ddot{t} + \frac{2GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1}\dot{r}\dot{t} = 0 \tag{20}$$

The symbols are

$$\Gamma^{t}_{ij} = \begin{bmatrix} 0 & \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

Pingbacks

Pingback: Maxwell's equations in cylindrical coordinates
Pingback: Covariant derivative in semi-log coordinates
Pingback: Christoffel symbols in sinusoidal coordinates
Pingback: Schwarzschild metric: acceleration
Pingback: Covariant derivative of a vector in the Schwarzschild metric
Pingback: Riemann tensor for an infinite plane of mass
Pingback: Riemann tensor in 2-d polar coordinates
Pingback: Riemann tensor in 2-d curved space
Pingback: Riemann tensor in the Schwarzschild metric
Pingback: Riemann tensor in 2-d flat space
Pingback: Ricci tensor and curvature scalar for a sphere
Pingback: Riemann tensor in a 2-d curved space
Pingback: Einstein equation for an exponential metric