

CHRISTOFFEL SYMBOLS: SYMMETRY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.3.

The Christoffel symbols are defined in terms of the basis vectors in a given coordinate system as:

$$(1) \quad \boxed{\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma_{ij}^k \mathbf{e}_k}$$

Remember that the basis vectors \mathbf{e}_i are defined so that

$$\begin{aligned} (2) \quad ds^2 &= ds \cdot ds \\ (3) \quad &= (dx^i \mathbf{e}_i) \cdot (dx^j \mathbf{e}_j) \\ (4) \quad &= \mathbf{e}_i \cdot \mathbf{e}_j dx^i dx^j \\ (5) \quad &\equiv g_{ij} dx^i dx^j \end{aligned}$$

In a locally flat frame using rectangular spatial coordinates, the basis vectors \mathbf{e}_i are all constants, so from 1, all the Christoffel symbols must be zero: $\Gamma_{ij}^k = 0$.

Now let's look at the second covariant derivative of a scalar field Φ :

$$\begin{aligned} (6) \quad \nabla_i \nabla_j \Phi &= \nabla_i (\partial_j \Phi) \\ (7) \quad &= \partial_i \partial_j \Phi - \Gamma_{ij}^k \partial_k \Phi \end{aligned}$$

where in 6 we used rule 1 for the covariant derivative: the covariant derivative of a scalar is the same as the ordinary derivative.

In the locally flat frame, this equation reduces to

$$(8) \quad \nabla_i \nabla_j \Phi = \partial_i \partial_j \Phi$$

Since the covariant derivative is a tensor, this is a tensor equation, and since ordinary partial derivatives commute, this equation is the same if we swap the indices i and j . Tensor equations must have the same form in all coordinate systems, so this implies that 7 must also be invariant if we

swap i and j . This means that the Christoffel symbols are symmetric under exchange of their two lower indices:

$$(9) \quad \boxed{\Gamma_{ij}^k = \Gamma_{ji}^k}$$

At first glance, this seems wrong, since from the definition 1 this symmetry implies that

$$(10) \quad \frac{\partial \mathbf{e}_i}{\partial x^j} = \frac{\partial \mathbf{e}_j}{\partial x^i}$$

In 2-D polar coordinates, if we take the usual unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ then both these vectors are constants as we change r and both of them change when we change θ , so it's certainly not true that $\partial \hat{\mathbf{r}} / \partial \theta = \partial \hat{\boldsymbol{\theta}} / \partial r$, for example. However, remember that the basis vectors we're using are *not* the usual unit vectors; rather they are defined so that condition 4 is true. In polar coordinates, we have

$$(11) \quad ds^2 = dr^2 + r^2 d\theta^2$$

so

$$(12) \quad \mathbf{e}_r = \hat{\mathbf{r}} = \cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}}$$

$$(13) \quad \mathbf{e}_\theta = r \hat{\boldsymbol{\theta}} = -r \sin \theta \hat{\mathbf{x}} + r \cos \theta \hat{\mathbf{y}}$$

For the derivatives, we have

$$(14) \quad \frac{\partial \mathbf{e}_r}{\partial \theta} = -\sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{y}} = \hat{\boldsymbol{\theta}}$$

$$(15) \quad \frac{\partial \mathbf{e}_\theta}{\partial r} = -\sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{y}} = \hat{\boldsymbol{\theta}}$$

Thus the condition 10 is actually satisfied here.

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