

CHRISTOFFEL SYMBOLS: SYMMETRY

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.3.

The Christoffel symbols are defined in terms of the basis vectors in a given coordinate system as:

$$\boxed{\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma_{ij}^k \mathbf{e}_k} \quad (1)$$

Remember that the basis vectors \mathbf{e}_i are defined so that

$$ds^2 = ds \cdot ds \quad (2)$$

$$= (dx^i \mathbf{e}_i) \cdot (dx^j \mathbf{e}_j) \quad (3)$$

$$= \mathbf{e}_i \cdot \mathbf{e}_j dx^i dx^j \quad (4)$$

$$\equiv g_{ij} dx^i dx^j \quad (5)$$

In a locally flat frame using rectangular spatial coordinates, the basis vectors \mathbf{e}_i are all constants, so from 1, all the Christoffel symbols must be zero: $\Gamma_{ij}^k = 0$.

Now let's look at the second covariant derivative of a scalar field Φ :

$$\nabla_i \nabla_j \Phi = \nabla_i (\partial_j \Phi) \quad (6)$$

$$= \partial_i \partial_j \Phi - \Gamma_{ij}^k \partial_k \Phi \quad (7)$$

where in 6 we used rule 1 for the covariant derivative: the covariant derivative of a scalar is the same as the ordinary derivative.

In the locally flat frame, this equation reduces to

$$\nabla_i \nabla_j \Phi = \partial_i \partial_j \Phi \quad (8)$$

Since the covariant derivative is a tensor, this is a tensor equation, and since ordinary partial derivatives commute, this equation is the same if we swap the indices i and j . Tensor equations must have the same form in all coordinate systems, so this implies that 7 must also be invariant if we

swap i and j . This means that the Christoffel symbols are symmetric under exchange of their two lower indices:

$$\boxed{\Gamma_{ij}^k = \Gamma_{ji}^k} \quad (9)$$

At first glance, this seems wrong, since from the definition 1 this symmetry implies that

$$\frac{\partial \mathbf{e}_i}{\partial x^j} = \frac{\partial \mathbf{e}_j}{\partial x^i} \quad (10)$$

In 2-D polar coordinates, if we take the usual unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ then both these vectors are constants as we change r and both of them change when we change θ , so it's certainly not true that $\partial \hat{\mathbf{r}} / \partial \theta = \partial \hat{\boldsymbol{\theta}} / \partial r$, for example. However, remember that the basis vectors we're using are *not* the usual unit vectors; rather they are defined so that condition 4 is true. In polar coordinates, we have

$$ds^2 = dr^2 + r^2 d\theta^2 \quad (11)$$

so

$$\mathbf{e}_r = \hat{\mathbf{r}} = \cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}} \quad (12)$$

$$\mathbf{e}_\theta = r \hat{\boldsymbol{\theta}} = -r \sin \theta \hat{\mathbf{x}} + r \cos \theta \hat{\mathbf{y}} \quad (13)$$

For the derivatives, we have

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = -\sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{y}} = \hat{\boldsymbol{\theta}} \quad (14)$$

$$\frac{\partial \mathbf{e}_\theta}{\partial r} = -\sin \theta \hat{\mathbf{x}} + \cos \theta \hat{\mathbf{y}} = \hat{\boldsymbol{\theta}} \quad (15)$$

Thus the condition 10 is actually satisfied here.

PINGBACKS

Pingback: [Christoffel symbols for Schwarzschild metric](#)

Pingback: [Christoffel symbols in terms of the metric tensor](#)

Pingback: [Gravitoelectric and gravitomagnetic densities](#)