CHRISTOFFEL SYMBOLS: SYMMETRY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.3.

The Christoffel symbols are defined in terms of the basis vectors in a given coordinate system as:

$$\boxed{\frac{\partial \mathbf{e}_i}{\partial x^j}=\Gamma_{ij}^k \mathbf{e}_k} \tag{1}$$

Remember that the basis vectors $\mathbf{e}_i$ are defined so that

$$\begin{aligned} ds^2 &= d\mathbf{s}\cdot d\mathbf{s} && (2)\\ &= \left(dx^i\mathbf{e}_i\right)\cdot\left(dx^j\mathbf{e}_j\right) && (3)\\ &= \mathbf{e}_i\cdot\mathbf{e}_j dx^i dx^j && (4)\\ &\equiv g_{ij}dx^i dx^j && (5)\end{aligned}$$

In a locally flat frame using rectangular spatial coordinates, the basis vectors $\mathbf{e}_i$ are all constants, so from 1, all the Christoffel symbols must be zero: $\Gamma_{ij}^k=0$.

Now let's look at the second covariant derivative of a scalar field Φ:

$$\begin{aligned} \nabla_i\nabla_j\Phi &= \nabla_i\left(\partial_j\Phi\right) && (6)\\ &= \partial_i\partial_j\Phi-\Gamma_{ij}^k\partial_k\Phi && (7)\end{aligned}$$

where in 6 we used rule 1 for the covariant derivate: the covariant derivative of a scalar is the same as the ordinary derivative.

In the locally flat frame, this equation reduces to

$$\nabla_i\nabla_j\Phi=\partial_i\partial_j\Phi \tag{8}$$

Since the covariant derivative is a tensor, this is a tensor equation, and since ordinary partial derivatives commute, this equation is the same if we swap the indices i and j. Tensor equations must have the same form in all coordinate systems, so this implies that 7 must also be invariant if we swap i and j. This means that the Christoffel symbols are symmetric under exchange of their two lower indices:

$$\boxed{\Gamma_{ij}^{k} = \Gamma_{ji}^{k}} \tag{9}$$

At first glance, this seems wrong, since from the definition 1 this symmetry implies that

$$\frac{\partial \mathbf{e}_i}{\partial x^j} = \frac{\partial \mathbf{e}_j}{\partial x^i} \tag{10}$$

In 2-D polar coordinates, if we take the usual unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$ then both these vectors are constants as we change r and both of them change when we change θ, so it's certainly not true that $\partial\hat{\mathbf{r}}/\partial\theta = \partial\hat{\theta}/\partial r$, for example. However, remember that the basis vectors we're using are *not* the usual unit vectors; rather they are defined so that condition 4 is true. In polar coordinates, we have

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{11}$$

so

$$\mathbf{e}_r = \hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} \tag{12}$$

$$\mathbf{e}_\theta = r\hat{\theta} = -r\sin\theta\hat{\mathbf{x}} + r\cos\theta\hat{\mathbf{y}} \tag{13}$$

For the derivatives, we have

$$\frac{\partial \mathbf{e}_r}{\partial\theta} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}} = \hat{\theta} \tag{14}$$

$$\frac{\partial \mathbf{e}_\theta}{\partial r} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}} = \hat{\theta} \tag{15}$$

Thus the condition 10 is actually satisfied here.

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