MAXWELL'S EQUATIONS IN CYLINDRICAL COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.3.

As an example of using the geodesic equation to calculate Christoffel symbols, we'll consider Maxwell's equations in cylindrical coordinates. We compare the geodesic equation:

$$g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j\dot{x}^i = 0 \tag{1}$$

with the expression for the Christoffel symbols:

$$\ddot{x}^m + \Gamma^m_{ij}\dot{x}^j\dot{x}^i = 0 \tag{2}$$

Using cylindrical coordinates to describe flat spacetime, we have

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 \tag{3}$$

The geodesic equation is, for each of the four coordinates

$$-\ddot{t} = 0 \tag{4}$$
$$\ddot{r} - \frac{1}{2}(2r)\dot{\theta}^2 = 0 \tag{5}$$
$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} = 0 \tag{6}$$
$$\ddot{z} = 0 \tag{7}$$

From this we get the Christoffel symbols:

$$\Gamma^{t}{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

$$\Gamma^{r}{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

$$\Gamma^{\theta}{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

$$\Gamma^{z}{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

The electromagnetic field tensor is, in rectangular coordinates:

$$F^{ij} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{bmatrix} \tag{12}$$

We can write this tensor in cylindrical coordinates as follows:

$$\left(F'\right)^{ij} = \begin{bmatrix} 0 & E_r & E_\theta & E_z \\ -E_r & 0 & B_z & -B_\theta \\ -E_\theta & -B_z & 0 & B_r \\ -E_z & B_\theta & -B_r & 0 \end{bmatrix} \tag{13}$$

Each entry in this tensor is found by the usual transformation rule. For example, since $t' = t$, $z' = z$, $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

$$E_r = \left(F'\right)^{tr} = F^{ij}\frac{\partial t}{\partial x'^i}\frac{\partial r}{\partial x'^j} \tag{14}$$

$$= F^{tx}\frac{x}{r} + F^{ty}\frac{y}{r} \tag{15}$$

$$= E_x\frac{x}{r} + E_y\frac{y}{r} \tag{16}$$

Using the same transformation rule, we get for the other 3 components:

$$E_{z'} = E_z \tag{17}$$

$$E_\theta = -E_x \frac{y}{r^2} + E_y \frac{x}{r^2} \tag{18}$$

Two of Maxwell's equationscan be written in terms of the electromagnetic tensor in rectangular coordinates as

$$\boxed{\partial_a F^{ba} = \frac{1}{\varepsilon_0} J^b} \tag{19}$$

This corresponds to the two Maxwell equations (in units where $c = 1/\sqrt{\mu_0 \varepsilon_0} = 1$):

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{20}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon_0} \mathbf{J} \tag{21}$$

In other coordinate systems, we can write 19 by replacing the ordinary derivative by the covariant derivative:

$$\boxed{\nabla_a F^{ba} = \frac{1}{\varepsilon_0} J^b} \tag{22}$$

To evaluate the covariant derivative, we write it in terms of the Christoffel symbols. Since there is a sum over the index a, we first write out the derivative without the sum:

$$\nabla_a F^{bc} = \partial_a F^{bc} + \Gamma^b_{\ aj} F^{jc} + \Gamma^c_{\ aj} F^{bj} \tag{23}$$

Now we can take the sum by setting $a = c$:

$$\nabla_a F^{ba} = \partial_a F^{ba} + \Gamma^b_{\ aj} F^{ja} + \Gamma^a_{\ aj} F^{bj} \tag{24}$$

Since $\Gamma^b_{\ aj} = \Gamma^b_{\ ja}$ and $F^{ja} = -F^{aj}$, the double sum $\Gamma^b_{\ aj} F^{ja}$ is always zero, so the second term on the RHS vanishes. We are left with

$$\nabla_a F^{ba} = \partial_a F^{ba} + \Gamma^a_{\ aj} F^{bj} \tag{25}$$

From 8 to 11, there are only 3 non-zero Γ^{b}_{aj}, so the expression on the RHS isn't terribly complicated when the sums are expanded. Considering the t, r, θ and z components in that order, we get

$$\partial_a F^{ta} + \Gamma^{a}_{aj} F^{tj} = \partial_r E_r + \partial_\theta E_\theta + \partial_z E_z + \frac{1}{r} E_r \tag{26}$$

$$\partial_a F^{ra} + \Gamma^{a}_{aj} F^{rj} = -\partial_t E_r + \partial_\theta B_z - \partial_z B_\theta \tag{27}$$

$$\partial_a F^{\theta a} + \Gamma^{a}_{aj} F^{\theta j} = -\partial_t E_\theta - \partial_r B_z + \partial_z B_r - \frac{1}{r} B_z \tag{28}$$

$$\partial_a F^{za} + \Gamma^{a}_{aj} F^{zj} = -\partial_t E_z + \partial_r B_\theta - \partial_\theta B_r + \frac{1}{r} B_\theta \tag{29}$$

As Moore points out, these are not the same components that we'd get if we converted Maxwell's equations to the usual cylindrical coordinate system in which the basis vectors are unit vectors, since in the cylindrical coordinate basis, the θ basis vector is $\mathbf{e}_\theta = r\hat{\theta}$. However, I did check that if you substitute 16 and 18 into 26 and work out all the derivatives, it is indeed true that

$$\partial_r E_r + \partial_\theta E_\theta + \partial_z E_z + \frac{1}{r} E_r = \partial_x E_x + \partial_y E_y + \partial_z E_z \tag{30}$$

Presumably the other 3 equations also work out to the 3 components of 21 if we work out the components of B_i in the cylindrical basis and then substitute them in and do the derivatives, though I'm too lazy to check this.

PINGBACKS

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