CHRISTOFFEL SYMBOLS IN TERMS OF THE METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.4.

It's time to find out how to calculate the Christoffel symbols. We start with their definition in terms of the basis vectors in some coordinate system:

$$\boxed{\partial_j \mathbf{e}_i = \Gamma^k_{\ ij}\mathbf{e}_k} \tag{1}$$

We take the scalar product of this equation with another basis vector $\mathbf{e}_l$ and use the definition of the metric tensor as $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$:

$$\begin{aligned}
\Gamma^k_{\ ij}\mathbf{e}_k \cdot \mathbf{e}_l &= \left(\partial_j \mathbf{e}_i\right) \cdot \mathbf{e}_l && (2)\\
\Gamma^k_{\ ij} g_{kl} &= \partial_j \left(\mathbf{e}_i \cdot \mathbf{e}_l\right) - \left(\partial_j \mathbf{e}_l\right) \cdot \mathbf{e}_i && (3)\\
&= \partial_j g_{il} - \Gamma^k_{\ lj}\mathbf{e}_k \cdot \mathbf{e}_i && (4)\\
&= \partial_j g_{il} - \Gamma^k_{\ lj} g_{ki} && (5)\\
\Gamma^k_{\ ij} g_{kl} + \Gamma^k_{\ lj} g_{ki} &= \partial_j g_{il} && (6)
\end{aligned}$$

In this equation the index k is a dummy (being summed over), so only the indices i, j and l are specified. We can cyclically permute these indices to generate two more equations:

$$\begin{aligned}
\Gamma^k_{\ jl} g_{ki} + \Gamma^k_{\ il} g_{kj} &= \partial_l g_{ji} && (7)\\
\Gamma^k_{\ li} g_{kj} + \Gamma^k_{\ ji} g_{kl} &= \partial_i g_{lj} && (8)
\end{aligned}$$

We can now use the symmetry of the Christoffel symbols to solve for $\Gamma^k_{\ ij}$ by swapping indices in 7 and 8 to get

$$\begin{aligned}
\Gamma^k_{\ lj} g_{ki} + \Gamma^k_{\ il} g_{kj} &= \partial_l g_{ji} && (9)\\
\Gamma^k_{\ il} g_{kj} + \Gamma^k_{\ ij} g_{kl} &= \partial_i g_{lj} && (10)
\end{aligned}$$

We can now add 6 to 10 and subtract 9 to get

$$2\Gamma^k_{\ ij} g_{kl} = \partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \tag{11}$$

Finally we can use the fact that

$$g^{ij}g_{jk} = \delta^i{}_k \tag{12}$$

and multiply both sides of 11 by g^{ml} to get

$$\begin{aligned}
2\Gamma^k_{ij}g_{kl}g^{ml} &= g^{ml}\left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) && (13)\\
\Gamma^k_{ij}\delta^m_k &= \frac{1}{2}g^{ml}\left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) && (14)\\
\Gamma^m_{ij} &= \frac{1}{2}g^{ml}\left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right) && (15)
\end{aligned}$$

This gives us a formula for explicitly evaluating Christoffel symbols:

$$\boxed{\Gamma^m_{ij} = \frac{1}{2}g^{ml}\left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}\right)} \tag{16}$$

This is a bit cumbersome to use as it requires finding the inverse metric tensor g^{ml} and has 3 sums over different derivatives.

Example. As an example, we'll work out Γ^m_{ij} for 2-D polar coordinates. The metric tensor and its inverse here are:

$$\begin{aligned}
g_{ij} &= \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} && (17)\\
g^{ij} &= \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix} && (18)
\end{aligned}$$

so the derivatives are

$$\begin{aligned}
\partial_r g_{ij} &= \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix} && (19)\\
\partial_\theta g_{ij} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} && (20)
\end{aligned}$$

Having only one non-zero derivative helps a lot, since the only non-zero term on the RHS of 16 is $\partial_r g_{\theta\theta} = 2r$. We'll work out a couple of the symbols explicitly and then give the final result.

$$\begin{aligned}
\Gamma^{\theta}_{r\theta} &= \frac{1}{2}g^{\theta l}\left(\partial_{\theta}g_{rl}+\partial_{r}g_{l\theta}-\partial_{l}g_{\theta r}\right) && (21)\\
&= \frac{1}{2}g^{\theta\theta}\left(\partial_{\theta}g_{r\theta}+\partial_{r}g_{\theta\theta}-\partial_{\theta}g_{\theta r}\right) && (22)\\
&= \frac{1}{2r^{2}}\left(2r\right) && (23)\\
&= \frac{1}{r} && (24)\\
&= \Gamma^{\theta}_{\theta r} && (25)\\
\Gamma^{r}_{\theta\theta} &= \frac{1}{2}g^{rl}\left(\partial_{\theta}g_{\theta l}+\partial_{\theta}g_{l\theta}-\partial_{l}g_{\theta\theta}\right) && (26)\\
&= \frac{1}{2}g^{rr}\left(\partial_{\theta}g_{\theta r}+\partial_{\theta}g_{r\theta}-\partial_{r}g_{\theta\theta}\right) && (27)\\
&= \frac{1}{2}(-2r) && (28)\\
&= -r && (29)
\end{aligned}$$

All of the other symbols are zero. The final results are

$$\begin{aligned}
\Gamma^{r}_{ij} &= \begin{bmatrix} 0 & 0\\ 0 & -r \end{bmatrix} && (30)\\
\Gamma^{\theta}_{ij} &= \begin{bmatrix} 0 & r^{-1}\\ r^{-1} & 0 \end{bmatrix} && (31)
\end{aligned}$$

Using these in 1 gives the 4 derivatives of the basis vectors:

$$\begin{aligned}
\partial_{r}\mathbf{e}_{r} &= \Gamma^{i}_{rr}\mathbf{e}_{i}=0 && (32)\\
\partial_{\theta}\mathbf{e}_{r} &= \Gamma^{i}_{r\theta}\mathbf{e}_{i}=\frac{1}{r}\mathbf{e}_{\theta} && (33)\\
\partial_{r}\mathbf{e}_{\theta} &= \Gamma^{i}_{\theta r}\mathbf{e}_{i}=\frac{1}{r}\mathbf{e}_{\theta} && (34)\\
\partial_{\theta}\mathbf{e}_{\theta} &= \Gamma^{i}_{\theta\theta}\mathbf{e}_{i}=-r\mathbf{e}_{r} && (35)
\end{aligned}$$

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