

## CHRISTOFFEL SYMBOLS IN TERMS OF THE METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.4.

It's time to find out how to calculate the Christoffel symbols. We start with their definition in terms of the basis vectors in some coordinate system:

$$\boxed{\partial_j \mathbf{e}_i = \Gamma_{ij}^k \mathbf{e}_k} \quad (1)$$

We take the scalar product of this equation with another basis vector  $\mathbf{e}_l$  and use the definition of the metric tensor as  $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ :

$$\Gamma_{ij}^k \mathbf{e}_k \cdot \mathbf{e}_l = (\partial_j \mathbf{e}_i) \cdot \mathbf{e}_l \quad (2)$$

$$\Gamma_{ij}^k g_{kl} = \partial_j (\mathbf{e}_i \cdot \mathbf{e}_l) - (\partial_j \mathbf{e}_l) \cdot \mathbf{e}_i \quad (3)$$

$$= \partial_j g_{il} - \Gamma_{lj}^k \mathbf{e}_k \cdot \mathbf{e}_i \quad (4)$$

$$= \partial_j g_{il} - \Gamma_{lj}^k g_{ki} \quad (5)$$

$$\Gamma_{ij}^k g_{kl} + \Gamma_{lj}^k g_{ki} = \partial_j g_{il} \quad (6)$$

In this equation the index  $k$  is a dummy (being summed over), so only the indices  $i, j$  and  $l$  are specified. We can cyclically permute these indices to generate two more equations:

$$\Gamma_{jl}^k g_{ki} + \Gamma_{il}^k g_{kj} = \partial_l g_{ji} \quad (7)$$

$$\Gamma_{li}^k g_{kj} + \Gamma_{ji}^k g_{kl} = \partial_i g_{lj} \quad (8)$$

We can now use the symmetry of the Christoffel symbols to solve for  $\Gamma_{ij}^k$  by swapping indices in 7 and 8 to get

$$\Gamma_{lj}^k g_{ki} + \Gamma_{il}^k g_{kj} = \partial_l g_{ji} \quad (9)$$

$$\Gamma_{il}^k g_{kj} + \Gamma_{ij}^k g_{kl} = \partial_i g_{lj} \quad (10)$$

We can now add 6 to 10 and subtract 9 to get

$$2\Gamma_{ij}^k g_{kl} = \partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \quad (11)$$

Finally we can use the fact that

$$g^{ij}g_{jk} = \delta^i_k \quad (12)$$

and multiply both sides of 11 by  $g^{ml}$  to get

$$2\Gamma^k_{ij}g_{kl}g^{ml} = g^{ml}(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}) \quad (13)$$

$$\Gamma^k_{ij}\delta^m_k = \frac{1}{2}g^{ml}(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}) \quad (14)$$

$$\Gamma^m_{ij} = \frac{1}{2}g^{ml}(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}) \quad (15)$$

This gives us a formula for explicitly evaluating Christoffel symbols:

$$\boxed{\Gamma^m_{ij} = \frac{1}{2}g^{ml}(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji})} \quad (16)$$

This is a bit cumbersome to use as it requires finding the inverse metric tensor  $g^{ml}$  and has 3 sums over different derivatives.

**Example.** As an example, we'll work out  $\Gamma^m_{ij}$  for 2-D polar coordinates. The metric tensor and its inverse here are:

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \quad (17)$$

$$g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix} \quad (18)$$

so the derivatives are

$$\partial_r g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix} \quad (19)$$

$$\partial_\theta g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (20)$$

Having only one non-zero derivative helps a lot, since the only non-zero term on the RHS of 16 is  $\partial_r g_{\theta\theta} = 2r$ . We'll work out a couple of the symbols explicitly and then give the final result.

$$\Gamma_{r\theta}^{\theta} = \frac{1}{2}g^{\theta l}(\partial_{\theta}g_{rl} + \partial_r g_{l\theta} - \partial_l g_{\theta r}) \quad (21)$$

$$= \frac{1}{2}g^{\theta\theta}(\partial_{\theta}g_{r\theta} + \partial_r g_{\theta\theta} - \partial_{\theta}g_{\theta r}) \quad (22)$$

$$= \frac{1}{2r^2}(2r) \quad (23)$$

$$= \frac{1}{r} \quad (24)$$

$$= \Gamma_{\theta r}^{\theta} \quad (25)$$

$$\Gamma_{\theta\theta}^r = \frac{1}{2}g^{rl}(\partial_{\theta}g_{\theta l} + \partial_{\theta}g_{l\theta} - \partial_l g_{\theta\theta}) \quad (26)$$

$$= \frac{1}{2}g^{rr}(\partial_{\theta}g_{\theta r} + \partial_{\theta}g_{r\theta} - \partial_r g_{\theta\theta}) \quad (27)$$

$$= \frac{1}{2}(-2r) \quad (28)$$

$$= -r \quad (29)$$

All of the other symbols are zero. The final results are

$$\Gamma_{ij}^r = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} \quad (30)$$

$$\Gamma_{ij}^{\theta} = \begin{bmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{bmatrix} \quad (31)$$

Using these in 1 gives the 4 derivatives of the basis vectors:

$$\partial_r \mathbf{e}_r = \Gamma_{rr}^i \mathbf{e}_i = 0 \quad (32)$$

$$\partial_{\theta} \mathbf{e}_r = \Gamma_{r\theta}^i \mathbf{e}_i = \frac{1}{r} \mathbf{e}_{\theta} \quad (33)$$

$$\partial_r \mathbf{e}_{\theta} = \Gamma_{\theta r}^i \mathbf{e}_i = \frac{1}{r} \mathbf{e}_{\theta} \quad (34)$$

$$\partial_{\theta} \mathbf{e}_{\theta} = \Gamma_{\theta\theta}^i \mathbf{e}_i = -r \mathbf{e}_r \quad (35)$$

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