

CHRISTOFFEL SYMBOLS IN TERMS OF THE METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.4.

It's time to find out how to calculate the Christoffel symbols. We start with their definition in terms of the basis vectors in some coordinate system:

$$(1) \quad \boxed{\partial_j \mathbf{e}_i = \Gamma_{ij}^k \mathbf{e}_k}$$

We take the scalar product of this equation with another basis vector \mathbf{e}_l and use the definition of the metric tensor as $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$:

$$(2) \quad \Gamma_{ij}^k \mathbf{e}_k \cdot \mathbf{e}_l = (\partial_j \mathbf{e}_i) \cdot \mathbf{e}_l$$

$$(3) \quad \Gamma_{ij}^k g_{kl} = \partial_j (\mathbf{e}_i \cdot \mathbf{e}_l) - (\partial_j \mathbf{e}_l) \cdot \mathbf{e}_i$$

$$(4) \quad = \partial_j g_{il} - \Gamma_{lj}^k \mathbf{e}_k \cdot \mathbf{e}_i$$

$$(5) \quad = \partial_j g_{il} - \Gamma_{lj}^k g_{ki}$$

$$(6) \quad \Gamma_{ij}^k g_{kl} + \Gamma_{lj}^k g_{ki} = \partial_j g_{il}$$

In this equation the index k is a dummy (being summed over), so only the indices i , j and l are specified. We can cyclically permute these indices to generate two more equations:

$$(7) \quad \Gamma_{jl}^k g_{ki} + \Gamma_{il}^k g_{kj} = \partial_l g_{ji}$$

$$(8) \quad \Gamma_{li}^k g_{kj} + \Gamma_{ji}^k g_{kl} = \partial_i g_{lj}$$

We can now use the symmetry of the Christoffel symbols to solve for Γ_{ij}^k by swapping indices in 7 and 8 to get

$$(9) \quad \Gamma_{lj}^k g_{ki} + \Gamma_{il}^k g_{kj} = \partial_l g_{ji}$$

$$(10) \quad \Gamma_{il}^k g_{kj} + \Gamma_{ij}^k g_{kl} = \partial_i g_{lj}$$

We can now add 6 to 10 and subtract 9 to get

$$(11) \quad 2\Gamma_{ij}^k g_{kl} = \partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}$$

Finally we can use the fact that

$$(12) \quad g^{ij} g_{jk} = \delta^i_k$$

and multiply both sides of 11 by g^{ml} to get

$$(13) \quad 2\Gamma_{ij}^k g_{kl} g^{ml} = g^{ml} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji})$$

$$(14) \quad \Gamma_{ij}^k \delta^m_k = \frac{1}{2} g^{ml} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji})$$

$$(15) \quad \Gamma_{ij}^m = \frac{1}{2} g^{ml} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji})$$

This gives us a formula for explicitly evaluating Christoffel symbols:

$$(16) \quad \boxed{\Gamma_{ij}^m = \frac{1}{2} g^{ml} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji})}$$

This is a bit cumbersome to use as it requires finding the inverse metric tensor g^{ml} and has 3 sums over different derivatives.

Example. As an example, we'll work out Γ_{ij}^m for 2-D polar coordinates. The metric tensor and its inverse here are:

$$(17) \quad g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

$$(18) \quad g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}$$

so the derivatives are

$$(19) \quad \partial_r g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix}$$

$$(20) \quad \partial_\theta g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Having only one non-zero derivative helps a lot, since the only non-zero term on the RHS of 16 is $\partial_r g_{\theta\theta} = 2r$. We'll work out a couple of the symbols explicitly and then give the final result.

$$(21) \quad \Gamma_{r\theta}^{\theta} = \frac{1}{2}g^{\theta l}(\partial_{\theta}g_{rl} + \partial_r g_{l\theta} - \partial_l g_{\theta r})$$

$$(22) \quad = \frac{1}{2}g^{\theta\theta}(\partial_{\theta}g_{r\theta} + \partial_r g_{\theta\theta} - \partial_{\theta}g_{\theta r})$$

$$(23) \quad = \frac{1}{2r^2}(2r)$$

$$(24) \quad = \frac{1}{r}$$

$$(25) \quad = \Gamma_{\theta r}^{\theta}$$

$$(26) \quad \Gamma_{\theta\theta}^r = \frac{1}{2}g^{rl}(\partial_{\theta}g_{\theta l} + \partial_{\theta}g_{l\theta} - \partial_l g_{\theta\theta})$$

$$(27) \quad = \frac{1}{2}g^{rr}(\partial_{\theta}g_{\theta r} + \partial_{\theta}g_{r\theta} - \partial_r g_{\theta\theta})$$

$$(28) \quad = \frac{1}{2}(-2r)$$

$$(29) \quad = -r$$

All of the other symbols are zero. The final results are

$$(30) \quad \Gamma_{ij}^r = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix}$$

$$(31) \quad \Gamma_{ij}^{\theta} = \begin{bmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{bmatrix}$$

Using these in 1 gives the 4 derivatives of the basis vectors:

$$(32) \quad \partial_r \mathbf{e}_r = \Gamma_{rr}^i \mathbf{e}_i = 0$$

$$(33) \quad \partial_{\theta} \mathbf{e}_r = \Gamma_{r\theta}^i \mathbf{e}_i = \frac{1}{r} \mathbf{e}_{\theta}$$

$$(34) \quad \partial_r \mathbf{e}_{\theta} = \Gamma_{\theta r}^i \mathbf{e}_i = \frac{1}{r} \mathbf{e}_{\theta}$$

$$(35) \quad \partial_{\theta} \mathbf{e}_{\theta} = \Gamma_{\theta\theta}^i \mathbf{e}_i = -r \mathbf{e}_r$$

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