CHRISTOFFEL SYMBOLS IN TERMS OF THE METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.4.

It's time to find out how to calculate the Christoffel symbols. We start with their definition in terms of the basis vectors in some coordinate system:

$$\boxed{\partial_j \mathbf{e}_i = \Gamma_{\ ij}^k \mathbf{e}_k} \tag{0.1}$$

We take the scalar product of this equation with another basis vector $\mathbf{e}_l$ and use the definition of the metric tensor as $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$:

$$\Gamma_{\ ij}^k \mathbf{e}_k \cdot \mathbf{e}_l = \left(\partial_j \mathbf{e}_i\right) \cdot \mathbf{e}_l \tag{0.2}$$

$$\Gamma_{\ ij}^k g_{kl} = \partial_j \left(\mathbf{e}_i \cdot \mathbf{e}_l\right) - \left(\partial_j \mathbf{e}_l\right) \cdot \mathbf{e}_i \tag{0.3}$$

$$= \partial_j g_{il} - \Gamma_{\ lj}^k \mathbf{e}_k \cdot \mathbf{e}_i \tag{0.4}$$

$$= \partial_j g_{il} - \Gamma_{\ lj}^k g_{ki} \tag{0.5}$$

$$\Gamma_{\ ij}^k g_{kl} + \Gamma_{\ lj}^k g_{ki} = \partial_j g_{il} \tag{0.6}$$

In this equation the index k is a dummy (being summed over), so only the indices i, j and l are specified. We can cyclically permute these indices to generate two more equations:

$$\Gamma_{\ jl}^k g_{ki} + \Gamma_{\ il}^k g_{kj} = \partial_l g_{ji} \tag{0.7}$$

$$\Gamma_{\ li}^k g_{kj} + \Gamma_{\ ji}^k g_{kl} = \partial_i g_{lj} \tag{0.8}$$

We can now use the symmetry of the Christoffel symbols to solve for $\Gamma_{\ ij}^k$ by swapping indices in 0.7 and 0.8 to get

$$\Gamma_{\ lj}^k g_{ki} + \Gamma_{\ il}^k g_{kj} = \partial_l g_{ji} \tag{0.9}$$

$$\Gamma_{\ il}^k g_{kj} + \Gamma_{\ ij}^k g_{kl} = \partial_i g_{lj} \tag{0.10}$$

We can now add 0.6 to 0.10 and subtract 0.9 to get

$$2\Gamma_{ij}^{k}g_{kl}=\partial_{j}g_{il}+\partial_{i}g_{lj}-\partial_{l}g_{ji} \tag{0.11}$$

Finally we can use the fact that

$$g^{ij}g_{jk}=\delta^{i}{}_{k} \tag{0.12}$$

and multiply both sides of 0.11 by g^{ml} to get

$$2\Gamma_{ij}^{k}g_{kl}g^{ml} = g^{ml}\left(\partial_{j}g_{il}+\partial_{i}g_{lj}-\partial_{l}g_{ji}\right) \tag{0.13}$$

$$\Gamma_{ij}^{k}\delta_{k}^{m} = \frac{1}{2}g^{ml}\left(\partial_{j}g_{il}+\partial_{i}g_{lj}-\partial_{l}g_{ji}\right) \tag{0.14}$$

$$\Gamma_{ij}^{m} = \frac{1}{2}g^{ml}\left(\partial_{j}g_{il}+\partial_{i}g_{lj}-\partial_{l}g_{ji}\right) \tag{0.15}$$

This gives us a formula for explicitly evaluating Christoffel symbols:

$$\boxed{\Gamma_{ij}^{m}=\frac{1}{2}g^{ml}\left(\partial_{j}g_{il}+\partial_{i}g_{lj}-\partial_{l}g_{ji}\right)} \tag{0.16}$$

This is a bit cumbersome to use as it requires finding the inverse metric tensor g^{ml} and has 3 sums over different derivatives.

Example. As an example, we'll work out Γ_{ij}^{m} for 2-D polar coordinates. The metric tensor and its inverse here are:

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{2} \end{bmatrix} \tag{0.17}$$

$$g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix} \tag{0.18}$$

so the derivatives are

$$\partial_{r}g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 2r \end{bmatrix} \tag{0.19}$$

$$\partial_{\theta}g_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{0.20}$$

Having only one non-zero derivative helps a lot, since the only non-zero term on the RHS of 0.16 is $\partial_{r}g_{\theta\theta}=2r$. We'll work out a couple of the symbols explicitly and then give the final result.

$$\Gamma^{\theta}_{r\theta} = \frac{1}{2}g^{\theta l}\left(\partial_{\theta}g_{rl}+\partial_{r}g_{l\theta}-\partial_{l}g_{\theta r}\right) \tag{0.21}$$
$$= \frac{1}{2}g^{\theta\theta}\left(\partial_{\theta}g_{r\theta}+\partial_{r}g_{\theta\theta}-\partial_{\theta}g_{\theta r}\right) \tag{0.22}$$
$$= \frac{1}{2r^{2}}(2r) \tag{0.23}$$
$$= \frac{1}{r} \tag{0.24}$$
$$= \Gamma^{\theta}_{\theta r} \tag{0.25}$$
$$\Gamma^{r}_{\theta\theta} = \frac{1}{2}g^{rl}\left(\partial_{\theta}g_{\theta l}+\partial_{\theta}g_{l\theta}-\partial_{l}g_{\theta\theta}\right) \tag{0.26}$$
$$= \frac{1}{2}g^{rr}\left(\partial_{\theta}g_{\theta r}+\partial_{\theta}g_{r\theta}-\partial_{r}g_{\theta\theta}\right) \tag{0.27}$$
$$= \frac{1}{2}(-2r) \tag{0.28}$$
$$= -r \tag{0.29}$$

All of the other symbols are zero. The final results are

$$\Gamma^{r}_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} \tag{0.30}$$
$$\Gamma^{\theta}_{ij} = \begin{bmatrix} 0 & r^{-1} \\ r^{-1} & 0 \end{bmatrix} \tag{0.31}$$

Using these in 0.1 gives the 4 derivatives of the basis vectors:

$$\partial_{r}\mathbf{e}_{r} = \Gamma^{i}{}_{rr}\mathbf{e}_{i} = 0 \tag{0.32}$$
$$\partial_{\theta}\mathbf{e}_{r} = \Gamma^{i}{}_{r\theta}\mathbf{e}_{i} = \frac{1}{r}\mathbf{e}_{\theta} \tag{0.33}$$
$$\partial_{r}\mathbf{e}_{\theta} = \Gamma^{i}{}_{\theta r}\mathbf{e}_{i} = \frac{1}{r}\mathbf{e}_{\theta} \tag{0.34}$$
$$\partial_{\theta}\mathbf{e}_{\theta} = \Gamma^{i}{}_{\theta\theta}\mathbf{e}_{i} = -r\mathbf{e}_{r} \tag{0.35}$$

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