

GEODESIC EQUATION IN TERMS OF CHRISTOFFEL SYMBOLS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.5.

The geodesic equation can be written in terms of Christoffel symbols. The geodesic equation in its original form is

$$\frac{d}{d\tau} \left(g_{aj} \frac{dx^j}{d\tau} \right) - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^a} \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0 \quad (1)$$

Working with the first term, we get (using a dot over a symbol to indicate the derivative with respect to τ):

$$\dot{g}_{aj} \dot{x}^j + g_{aj} \ddot{x}^j - \frac{1}{2} \partial_a g_{ij} \dot{x}^i \dot{x}^j = 0 \quad (2)$$

$$\frac{1}{2} \dot{g}_{aj} \dot{x}^j + \frac{1}{2} \dot{g}_{aj} \dot{x}^j - \frac{1}{2} \partial_a g_{ij} \dot{x}^i \dot{x}^j + g_{aj} \ddot{x}^j = 0 \quad (3)$$

$$\frac{1}{2} [\partial_i g_{aj} \dot{x}^j \dot{x}^i + \partial_i g_{aj} \dot{x}^j \dot{x}^i - \partial_a g_{ij} \dot{x}^i \dot{x}^j] + g_{aj} \ddot{x}^j = 0 \quad (4)$$

$$\frac{1}{2} [\partial_i g_{aj} \dot{x}^j \dot{x}^i + \partial_j g_{ai} \dot{x}^j \dot{x}^i - \partial_a g_{ij} \dot{x}^i \dot{x}^j] + g_{aj} \ddot{x}^j = 0 \quad (5)$$

$$\frac{1}{2} g^{ma} [\partial_i g_{aj} + \partial_j g_{ai} - \partial_a g_{ij}] \dot{x}^j \dot{x}^i + g^{ma} g_{aj} \ddot{x}^j = 0 \quad (6)$$

$$\Gamma_{ij}^m \dot{x}^j \dot{x}^i + \delta_j^m \ddot{x}^j = 0 \quad (7)$$

$$\ddot{x}^m + \Gamma_{ij}^m \dot{x}^j \dot{x}^i = 0 \quad (8)$$

In 5 we swapped the dummy indices i and j in the second term in the brackets. In 6 we multiplied through by g^{ma} and in 7 we used the expression for the Christoffel symbols in terms of the metric.

$$\Gamma_{ij}^m = \frac{1}{2} g^{ml} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}) \quad (9)$$

Thus the equation

$$\boxed{\ddot{x}^m + \Gamma_{ij}^m \dot{x}^j \dot{x}^i = 0} \quad (10)$$

is formally equivalent to the geodesic equation.

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