GEODESIC EQUATION IN TERMS OF CHRISTOFFEL SYMBOLS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.5.

The geodesic equation can be written in terms of Christoffel symbols. The geodesic equation in its original form is

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right)-\frac{1}{2}\frac{\partial g_{ij}}{\partial x^a}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}=0 \tag{1}$$

Working with the first term, we get (using a dot over a symbol to indicate the derivative with respect to τ):

$$\dot{g}_{aj}\dot{x}^j+g_{aj}\ddot{x}^j-\frac{1}{2}\partial_a g_{ij}\dot{x}^i\dot{x}^j=0 \tag{2}$$

$$\frac{1}{2}\dot{g}_{aj}\dot{x}^j+\frac{1}{2}\dot{g}_{aj}\dot{x}^j-\frac{1}{2}\partial_a g_{ij}\dot{x}^i\dot{x}^j+g_{aj}\ddot{x}^j=0 \tag{3}$$

$$\frac{1}{2}\left[\partial_i g_{aj}\dot{x}^j\dot{x}^i+\partial_i g_{aj}\dot{x}^j\dot{x}^i-\partial_a g_{ij}\dot{x}^i\dot{x}^j\right]+g_{aj}\ddot{x}^j=0 \tag{4}$$

$$\frac{1}{2}\left[\partial_i g_{aj}\dot{x}^j\dot{x}^i+\partial_j g_{ai}\dot{x}^j\dot{x}^i-\partial_a g_{ij}\dot{x}^i\dot{x}^j\right]+g_{aj}\ddot{x}^j=0 \tag{5}$$

$$\frac{1}{2}g^{ma}\left[\partial_i g_{aj}+\partial_j g_{ai}-\partial_a g_{ij}\right]\dot{x}^j\dot{x}^i+g^{ma}g_{aj}\ddot{x}^j=0 \tag{6}$$

$$\Gamma^m_{ij}\dot{x}^j\dot{x}^i+\delta^m_j\ddot{x}^j=0 \tag{7}$$

$$\ddot{x}^m+\Gamma^m_{ij}\dot{x}^j\dot{x}^i=0 \tag{8}$$

In 5 we swapped the dummy indices i and j in the second term in the brackets. In 6 we multiplied through by g^{ma} and in 7 we used the expression for the Christoffel symbols in terms of the metric.

$$\Gamma^m_{ij}=\frac{1}{2}g^{ml}\left(\partial_j g_{il}+\partial_i g_{lj}-\partial_l g_{ji}\right) \tag{9}$$

Thus the equation

$$\boxed{\ddot{x}^m+\Gamma^m_{ij}\dot{x}^j\dot{x}^i=0} \tag{10}$$

is formally equivalent to the geodesic equation.

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