

COVARIANT DERIVATIVE IN SEMI-LOG COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.5.

As another example of using the geodesic equation to calculate Christoffel symbols, we'll consider the semi-log coordinates introduced earlier:

$$(1) \quad p = x$$
$$(2) \quad q = e^{by}$$

The invariant interval is

$$(3) \quad ds^2 = dx^2 + dy^2$$
$$(4) \quad = dp^2 + \frac{1}{(bq)^2} dq^2$$

so the metric is

$$(5) \quad g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{(bq)^2} \end{bmatrix}$$

We compare the geodesic equation:

$$(6) \quad g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2} \partial_a g_{ij} \right) \dot{x}^j \dot{x}^i = 0$$

with the expression for the Christoffel symbols:

$$(7) \quad \ddot{x}^m + \Gamma_{ij}^m \dot{x}^j \dot{x}^i = 0$$

We get

$$(8) \quad \ddot{p} = 0$$

$$(9) \quad \frac{1}{(bq)^2} \ddot{q} - \frac{2}{b^2 q^3} \dot{q}^2 - \frac{1}{2} \left(-\frac{2}{b^2 q^3} \dot{q}^2 \right) = 0$$

$$(10) \quad \ddot{q} - \frac{1}{q} \dot{q}^2 = 0$$

The Christoffel symbols are thus

$$(11) \quad \Gamma_{ij}^p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(12) \quad \Gamma_{ij}^q = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{q} \end{bmatrix}$$

Now consider a vector field $A^i = [0, Cx]$ (where C is a constant) in rectangular coordinates. In these coordinates, its covariant derivative is just the normal derivative, so

$$(13) \quad \nabla_i A^j = \partial_i A^j$$

$$(14) \quad = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$$

In the semi-log system, we have

$$(15) \quad \nabla_i A^j = \partial_i A^j + \Gamma_{ik}^j A^k$$

To use this formula we need A^i in the pq system, which we can get by the usual transformation, using:

$$(16) \quad A^p = A^i \partial_i p = A^x = 0$$

$$(17) \quad A^q = A^i \partial_i q = A^y (be^{by}) = Cbpq$$

With these transformations, we get

$$(18) \quad \nabla_i A^j = \begin{bmatrix} 0 & Cbq \\ 0 & Cbp - Cbp \end{bmatrix}$$

$$(19) \quad = \begin{bmatrix} 0 & Cbq \\ 0 & 0 \end{bmatrix}$$

The difference in gradients is because of the different scales used in the vertical direction. Consider first the vertical component of A . In rectangular

coordinates this has a constant value for a given value of x namely $A^y = Cx$. However, if we use q to measure vertical distance, a unit change in y results in a larger and larger change in q the higher up the vertical axis we go, so A^q must increase as q increases, even if $x = p$ is held constant. This is reflected by 17, where A^q is proportional to q as well as p .

If we used ordinary derivatives to calculate the gradient of A in the pq system, we would get $\partial_q A^q = Cbp$. However, the 'true' value of the vertical component of A doesn't change as we move up or down a vertical line, and this is reflected by the $\Gamma_{ik}^q A^k = -Cbp$ correction term that is present in the covariant derivative 15, with the result that $\nabla_q A^q = \partial_q A^q + \Gamma_{qk}^q A^k = Cbp - Cbp = 0$.

The value of $\nabla_p A^q = Cbq$ again reflects the fact that a vertical component that is constant in rectangular coordinates must get numerically larger with increasing height in the pq system.

As a final test that all is well, we can use the standard tensor transformation to transform 19 back to rectangular coordinates, where we denote rectangular coordinates by r^i and the semi-log pq system by s :

$$(20) \quad [\nabla_i A^j]_r = [\nabla_a A^b]_s \frac{\partial s^a}{\partial r^i} \frac{\partial r^j}{\partial s^b}$$

The only non-zero component of $[\nabla_a A^b]_s$ is $[\nabla_p A^q]_s = Cbq$, so the RHS has only one non-zero term, which occurs when $a = p$, $i = x$, $b = q$ and $j = y$. For this term we have

$$(21) \quad [\nabla_x A^y]_r = Cbq \frac{\partial p}{\partial x} \frac{\partial y}{\partial q}$$

$$(22) \quad = Cbq (1) \frac{1}{bq}$$

$$(23) \quad = C$$

The overall transformation thus gives 14 back again.