CHRISTOFFEL SYMBOLS IN SINUSOIDAL COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.6.

We are now in a position to revisit the system with sinusoidal coordinates. To review, we had a 2-d system with coordinates u and w defined in terms of the usual rectangular coordinates x and y by

$$u = x \tag{1}$$
$$w = y - A\sin(bx) \tag{2}$$

The metric for this system is

$$g_{ij} = \begin{bmatrix} 1 + [Ab\cos(bu)]^2 & Ab\cos(bu) \\ Ab\cos(bu) & 1 \end{bmatrix} \tag{3}$$

We looked at an object with a velocity given by $\mathbf{v} = [v, 0]$ where v is a constant. Clearly the acceleration of the object is zero, but if we calculate the velocity components in the uw system, we get

$$v^u = v \tag{4}$$
$$v^w = -Ab\cos(bu)\,v \tag{5}$$

so although $dv^u/dt = 0$, $dv^w/dt \neq 0$ since $u = x = vt$ varies with time.

To get the 'true' acceleration, we need to find the actual differential $d\mathbf{v}$ and divide this by dt. We've seen how to do this when we defined the Christoffel symbols:

$$d\mathbf{v} = \left[\frac{\partial v^k}{\partial x^j} + v^i \Gamma^k_{\ ij}\right] \mathbf{e}_k dx^j \tag{6}$$

To calculate $d\mathbf{v}$, we need the $\Gamma^k_{\ ij}$ for the uw system, which we can calculate in the usual way using the geodesic equation. We start with

$$g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j \dot{x}^i = 0 \tag{7}$$

Since g_{ij} is independent of w only derivatives with respect to u are non-zero. Consider first $a = u$; then we get

$$\left[1+[Ab\cos(bu)]^2\right]\ddot{u}+(Ab\cos bu)\,\ddot{w}-A^2b^3\cos bu\sin bu\dot{u}^2+\left[-Ab^2\sin bu-\left(-Ab^2\sin bu\right)\right]\dot{u}\dot{v}=0 \tag{8}$$

$$\left[1+[Ab\cos(bu)]^2\right]\ddot{u}+(Ab\cos bu)\,\ddot{w}-A^2b^3\cos bu\sin bu\dot{u}^2=0 \tag{9}$$

Now for $a = w$:

$$Ab\cos bu\ddot{u}+\ddot{w}-Ab^2\sin bu\dot{u}^2=0 \tag{10}$$

We would like to compare these equations with the equation involving the Christoffel symbols:

$$\ddot{x}^m+\Gamma^m_{ij}\dot{x}^j\dot{x}^i=0 \tag{11}$$

However, because the metric here is not diagonal, we get second derivatives of more than one coordinate in each equation. We can eliminate $\ddot{w}$ by multiplying 10 by $Ab\cos bu$ and subtracting it from 9. This gives the convenient result:

$$\ddot{u}=0 \tag{12}$$

From this we conclude that

$$\Gamma^u_{ij}=\begin{bmatrix}0&0\\0&0\end{bmatrix} \tag{13}$$

We can substitute 12 into 10 to get

$$\ddot{w}-Ab^2\sin bu\dot{u}^2=0 \tag{14}$$

from which we conclude

$$\Gamma^w_{ij}=\begin{bmatrix}-Ab^2\sin bu&0\\0&0\end{bmatrix} \tag{15}$$

We can now evaluate 6 to find:

$$
\begin{aligned}
dv^u &= \left[\frac{\partial v^u}{\partial x^j} + v^i \Gamma^u_{ij}\right] dx^j & (16)\\
&= 0 & (17)\\
dv^w &= \left[\frac{\partial v^w}{\partial x^j} + v^i \Gamma^w_{ij}\right] dx^j & (18)\\
&= \left[Ab^2 v \sin bu - vAb^2 \sin bu\right] du & (19)\\
&= 0 & (20)
\end{aligned}
$$

Thus by taking the proper derivative using the Christoffel symbols, the differentials of both components of velocity are zero, so the acceleration is zero in the *uw* system as well.