

CHRISTOFFEL SYMBOLS IN SINUSOIDAL COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.6.

We are now in a position to revisit the system with sinusoidal coordinates. To review, we had a 2-d system with coordinates u and w defined in terms of the usual rectangular coordinates x and y by

$$u = x \tag{1}$$

$$w = y - A \sin(bx) \tag{2}$$

The metric for this system is

$$g_{ij} = \begin{bmatrix} 1 + [Ab \cos(bu)]^2 & Ab \cos(bu) \\ Ab \cos(bu) & 1 \end{bmatrix} \tag{3}$$

We looked at an object with a velocity given by $\mathbf{v} = [v, 0]$ where v is a constant. Clearly the acceleration of the object is zero, but if we calculate the velocity components in the uw system, we get

$$v^u = v \tag{4}$$

$$v^w = -Ab \cos(bu) v \tag{5}$$

so although $dv^u/dt = 0$, $dv^w/dt \neq 0$ since $u = x = vt$ varies with time.

To get the 'true' acceleration, we need to find the actual differential $d\mathbf{v}$ and divide this by dt . We've seen how to do this when we defined the Christoffel symbols:

$$d\mathbf{v} = \left[\frac{\partial v^k}{\partial x^j} + v^i \Gamma_{ij}^k \right] \mathbf{e}_k dx^j \tag{6}$$

To calculate $d\mathbf{v}$, we need the Γ_{ij}^k for the uw system, which we can calculate in the usual way using the geodesic equation. We start with

$$g_{aj} \ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2} \partial_a g_{ij} \right) \dot{x}^j \dot{x}^i = 0 \tag{7}$$

Since g_{ij} is independent of w only derivatives with respect to u are non-zero. Consider first $a = u$; then we get

$$\left[1 + [Ab \cos(bu)]^2\right] \ddot{u} + (Ab \cos bu) \ddot{w} - A^2 b^3 \cos bu \sin bu \dot{u}^2 + [-Ab^2 \sin bu - (-Ab^2 \sin bu)] \dot{u} \dot{v} = 0 \quad (8)$$

$$\left[1 + [Ab \cos(bu)]^2\right] \ddot{u} + (Ab \cos bu) \ddot{w} - A^2 b^3 \cos bu \sin bu \dot{u}^2 = 0 \quad (9)$$

Now for $a = w$:

$$Ab \cos bu \ddot{u} + \ddot{w} - Ab^2 \sin bu \dot{u}^2 = 0 \quad (10)$$

We would like to compare these equations with the equation involving the Christoffel symbols:

$$\ddot{x}^m + \Gamma_{ij}^m \dot{x}^j \dot{x}^i = 0 \quad (11)$$

However, because the metric here is not diagonal, we get second derivatives of more than one coordinate in each equation. We can eliminate \ddot{w} by multiplying 10 by $Ab \cos bu$ and subtracting it from 9. This gives the convenient result:

$$\ddot{u} = 0 \quad (12)$$

From this we conclude that

$$\Gamma_{ij}^u = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (13)$$

We can substitute 12 into 10 to get

$$\ddot{w} - Ab^2 \sin bu \dot{u}^2 = 0 \quad (14)$$

from which we conclude

$$\Gamma_{ij}^w = \begin{bmatrix} -Ab^2 \sin bu & 0 \\ 0 & 0 \end{bmatrix} \quad (15)$$

We can now evaluate 6 to find:

$$dv^u = \left[\frac{\partial v^u}{\partial x^j} + v^i \Gamma_{ij}^u \right] dx^j \quad (16)$$

$$= 0 \quad (17)$$

$$dv^w = \left[\frac{\partial v^w}{\partial x^j} + v^i \Gamma_{ij}^w \right] dx^j \quad (18)$$

$$= [Ab^2 v \sin bu - v Ab^2 \sin bu] du \quad (19)$$

$$= 0 \quad (20)$$

Thus by taking the proper derivative using the Christoffel symbols, the differentials of both components of velocity are zero, so the acceleration is zero in the uw system as well.