LOCALLY INERTIAL FRAMES (LIFS)

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Box 17.7.

We've seen that for a smooth metric, we can define a locally flat coordinate system at a given event in which the metric is the flat spacetime metric η_{ij}. Although we specify the value of the metric tensor at that point, we haven't imposed any conditions on its derivatives with respect to the various coordinates. If we impose the addition constraint that all the derivatives of the metric are zero, then we have a *locally inertial frame* or LIF.

It's not immediately obvious that we *can* impose this restriction, so let's see how we can demonstrate this. We start with some arbitrary metric g_{ij} which doesn't satisfy the LIF conditions and consider the transformation to another metric g'_{ij} on which we try to impose these conditions. As usual, the transformation is

$$g'_{ij} = \partial'_i x^a \partial'_j x^b g_{ab} \tag{1}$$

Each of the partial derivatives is a function of the primed coordinates so, for a region close to the event point P, we can expand these derivatives in Taylor series:

$$\partial'_i x^a \left(x'_P + \Delta x'\right) = a^a_i + b^a_{ij} \Delta x'^j + c^a_{ijk} \Delta x'^j \Delta x'^k + \ldots \tag{2}$$

where the coefficients are defined in terms of $\partial'_i x^a$ and its derivatives, all evaluated at P:

$$\begin{aligned}
a^a_i &= \partial'_i x^a \left(x'_P\right) &(3)\\
b^a_{ij} &= \partial'_j \partial'_i x^a \left(x'_P\right) &(4)\\
c^a_{ijk} &= \frac{1}{2} \partial'_k \partial'_j \partial'_i x^a \left(x'_P\right) &(5)
\end{aligned}$$

We can also think of the unprimed metric as a function of the primed coordinates, and expand it in a Taylor series as well:

$$g_{ab}\left(x'_P+\Delta x'\right)=g_{ab}\left(x'_P\right)+\Delta x'^c\partial'_c g_{ab}\left(x'_P\right)+\frac{1}{2}\Delta x'^d\Delta x'^c\partial'_d\partial'_c g_{ab}\left(x'_P\right)+\ldots \tag{6}$$

We can now substitute these two series into 1 and collect terms. Due to the large number of indices floating about, it's easier to use a condensed notation for this step. We can temporarily drop the indices to get

$$\partial' x = a+b\Delta x'+c\left(\Delta x'\right)^2 \tag{7}$$

$$g = g_P+\Delta x'\partial' g_P+\frac{1}{2}\left(\Delta x'\right)^2\partial'^2 g_P \tag{8}$$

The transformation now becomes, up to second order terms:

$$g'\left(x'_P+\Delta x'\right)=\left(\partial' x\right)^2 g \tag{9}$$

$$=\left(a+b\Delta x'+c\left(\Delta x'\right)^2\right)\left(a+b\Delta x'+c\left(\Delta x'\right)^2\right)\left(g_P+\Delta x'\partial' g_P+\frac{1}{2}\left(\Delta x'\right)^2\partial'^2 g_P\right) \tag{10}$$

$$=a^2 g_P+\Delta x'\left[abg_P+bag_P+a^2\partial' g_P\right]+ \left(\Delta x'\right)^2\left[\frac{1}{2}a^2\partial'^2 g_P+ab\partial' g_P+acg_P+ba\partial' g_P+b^2 g_P+cag_P\right] \tag{11}$$

The last equation is the Taylor expansion of $g'\left(x'_P+\Delta x'\right)$ where every factor is now a function of x'. That is

$$g'_{ij}\left(x'_P+\Delta x'\right)=g'_{ij}\left(x'_P\right)+\Delta x'^k\partial'_k g'_{ij}\left(x'_P\right)+\Delta x'^\ell\Delta x'^k\partial'_\ell\partial'_k g'_{ij}\left(x'_P\right)$$

We therefore have, taking each term separately and restoring the indices:

$$g'_{ij}\left(x'_P\right) = a^2 g_P \tag{12}$$
$$= a_i^a a_j^b \left(g_{ab}\right)_P \tag{13}$$
$$\partial'_k g'_{ij}\left(x'_P\right) = abg_P + bag_P + a^2\partial' g_P \tag{14}$$
$$= \left[a_i^a b_{jk}^b + b_{ik}^a a_j^b\right]\left(g_{ab}\right)_P + a_i^a a_j^b \partial'_k \left(g_{ab}\right)_P \tag{15}$$
$$\partial'_\ell \partial'_k g'_{ij}\left(x'_P\right) = \frac{1}{2}a^2\partial'^2 g_P + ab\partial' g_P + acg_P + ba\partial' g_P + b^2 g_P + cag_P \tag{16}$$
$$= \frac{1}{2}a_i^a a_j^b \partial'_k \partial'_\ell \left(g_{ab}\right)_P + a_i^a b_{jk}^b \partial'_\ell \left(g_{ab}\right)_P + a_i^a c_{jk\ell}^b \left(g_{ab}\right)_P +$$
$$b_{ik}^a a_j^b \partial'_\ell \left(g_{ab}\right)_P + b_{ik}^a b_{j\ell}^b \left(g_{ab}\right)_P + c_{ik\ell}^a a_j^b \left(g_{ab}\right)_P \tag{17}$$

We won't actually try to solve these equations, but what we need to do is see if we have enough freedom to impose the conditions above, that is, that $g'_{ij}\left(x'_P\right) = \eta_{ij}$ and $\partial'_k g'_{ij}\left(x'_P\right) = 0$. To do this, we need to see how many independent equations each of these equations provides.

Consider 13 first. Each index i and j has 4 possible values, but because $g'_{ij} = g'_{ji}$ we have to exclude permutations of i and j. We can get these using binomial coefficients:

$$N = \binom{4}{1} + \binom{4}{2} = 4 + 6 = 10$$

For 15, the symmetry of g' means we have $4 \times 10 = 40$ (10 choices for i and j, times the 4 different values of k) independent equations. Finally, for 17, since the order of partial derivatives doesn't matter and the metric is symmetric, we have $10 \times 10 = 100$ independent equations.

The original metric g_{ij} is assumed to be given, so we can't change that, so how many variables can be solve for in these equations? The transformation is effectively determined (up to second order) by the coefficients a_i^a, b_{ij}^a and c_{ijk}^a, so we need to figure out how many of these are independent of each other.

For a_i^a every choice of the two indices gives an independent quantity, so there are 16 different variables. Thus we have 6 more degrees of freedom than we need to solve 13 and set $g'_{ij} = \eta_{ij}$.

For b_{ij}^a, the order of the partial derivatives doesn't matter so there are 10 possible choices of i and j for each value of a, giving $10 \times 4 = 40$. Having chosen the a_i^a in solving 13, we can plug these into the 40 equations specified by 15 and solve for the 40 b_{ij}^a coefficients. Therefore, in principle, we *can* set $\partial'_k g'_{ij}\left(x'_P\right) = 0$.

When it comes to c^a_{ijk}, again the order of the 3 partial derivatives doesn't matter, so for each value of a, we can have $i=j=k$, or two indices equal with the third different, or all three indices different. The total number of choices for each value of a is then

$$N_c=\binom{4}{1}+2\binom{4}{2}+\binom{4}{3}=20$$

To get the middle term, note that $\binom{4}{2}$ is the number of ways of choosing 2 numbers out of 4, but it assumes there are only 2 slots into which these numbers can be placed. In our case, one of the numbers is used twice, so this term would imply that, for example, 1-1-2 and 2-2-1 are just permutations of the same combination, whereas we want them to be distinct, so we have to multiply by 2. The total number of independent c^a_{ijk} is therefore $4\times 20=80$. The fact that there are 100 independent equations specified by 17 means that we can't impose values on all of the $\partial'_\ell\partial'_k g'_{ij}(x'_P)$ so in general even if we require 80 of them to be zero, the other 20 may have non-zero values. It turns out that this is a result of the inherent curvature of the spacetime, so it's not surprising that we can't get rid of that.

The LIF imposes stricter conditions on the local metric that the locally flat frame we mentioned at the start. The latter just specifies the metric at a given point, while the LIF specifies both the metric and its derivatives.

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