

SCHWARZSCHILD METRIC: ACCELERATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.7.

An object's four-acceleration \mathbf{a} is defined as

$$\mathbf{a} \equiv \frac{d\mathbf{u}}{d\tau} \quad (1)$$

For an object at rest in the Schwarzschild (S) frame, the four-velocity is given by

$$\mathbf{u} = \left[\left(1 - \frac{2GM}{r} \right)^{-1/2}, 0, 0, 0 \right] \quad (2)$$

To find the acceleration, we need to take the covariant derivative of \mathbf{u} :

$$d\mathbf{u} = \left[\frac{\partial u^k}{\partial x^j} + u^i \Gamma_{ij}^k \right] \mathbf{e}_k dx^j \quad (3)$$

The individual components of $d\mathbf{u}$ are then

$$du^k = \left[\frac{\partial u^k}{\partial x^j} + u^i \Gamma_{ij}^k \right] dx^j \quad (4)$$

The partial derivative term is non-zero only for $k = t$ and $j = r$. Since the object is at rest, the only non-zero component of u^i is u^t , and the only non-zero differential is dx^t . Therefore, the only Christoffel symbols that we need to consider are Γ_{tt}^k .

The Christoffel symbols for the S metric are, for $k = t$:

$$\Gamma_{ij}^t = \begin{bmatrix} 0 & \frac{GM}{r^2} \left(1 - \frac{2GM}{r} \right)^{-1} & 0 & 0 \\ \frac{GM}{r^2} \left(1 - \frac{2GM}{r} \right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (5)$$

Since $\Gamma_{tt}^t = 0$, we conclude that $a^t = 0$.

Now consider $k = r$. The partial derivative is zero, but the second term is not, since

$$\Gamma_{ij}^r = \begin{bmatrix} \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r \left(1 - \frac{2GM}{r}\right) & 0 \\ 0 & 0 & 0 & -r \sin^2 \theta \left(1 - \frac{2GM}{r}\right) \end{bmatrix} \quad (6)$$

If $i = j = t$ we get a non-zero term:

$$du^r = u^t \Gamma_{tt}^r dx^t \quad (7)$$

$$= \left(1 - \frac{2GM}{r}\right)^{-1/2} \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right) dt \quad (8)$$

$$= \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{1/2} dt \quad (9)$$

Now we divide both sides by $d\tau$ and use the relation between S time and proper time for an object at rest:

$$\frac{dt}{d\tau} = \left(1 - \frac{2GM}{r}\right)^{-1/2} \quad (10)$$

$$a^r = \frac{du^r}{d\tau} = \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{1/2} \left(1 - \frac{2GM}{r}\right)^{-1/2} = \frac{GM}{r^2} \quad (11)$$

The remaining two components a^θ and a^ϕ are both zero, since $\Gamma_{tt}^\theta = \Gamma_{tt}^\phi = 0$. Thus

$$\mathbf{a} = \left[0, \frac{GM}{r^2}, 0, 0\right] \quad (12)$$

The magnitude can be found from the S metric:

$$ds^2 = - \left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (13)$$

We get

$$a = \sqrt{\mathbf{a} \cdot \mathbf{a}} \quad (14)$$

$$= g_{ij} a^i a^j \quad (15)$$

$$= \sqrt{g_{rr}} \frac{GM}{r^2} \quad (16)$$

$$= \frac{GM}{r^2 \sqrt{1 - 2GM/r}} \quad (17)$$

This is the acceleration you would need to remain at rest at a radial distance r . Note that the acceleration becomes infinite as we approach $r = 2GM$, showing that it's impossible to remain at rest at the event horizon.