SCHWARZSCHILD METRIC: ACCELERATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.7.

An object's four-acceleration $\mathbf{a}$ is defined as

$$\mathbf{a}\equiv\frac{d\mathbf{u}}{d\tau} \tag{0.1}$$

For an object at rest in the Schwarzschild (S) frame, the four-velocity is given by

$$\mathbf{u}=\left[\left(1-\frac{2GM}{r}\right)^{-1/2},0,0,0\right] \tag{0.2}$$

To find the acceleration, we need to take the covariant derivative of $\mathbf{u}$:

$$d\mathbf{u}=\left[\frac{\partial u^k}{\partial x^j}+u^i\Gamma^k_{\ ij}\right]\mathbf{e}_k dx^j \tag{0.3}$$

The individual components of $d\mathbf{u}$ are then

$$du^k=\left[\frac{\partial u^k}{\partial x^j}+u^i\Gamma^k_{\ ij}\right]dx^j \tag{0.4}$$

The partial derivative term is non-zero only for $k=t$ and $j=r$. Since the object is at rest, the only non-zero component of u^i is u^t, and the only non-zero differential is dx^t. Therefore, the only Christoffel symbols that we need to consider are $\Gamma^k_{\ tt}$.

The Christoffel symbols for the S metric are, for $k=t$:

$$\Gamma^t_{\ ij}=\left[\begin{array}{cccc}0 & \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \tag{0.5}$$

Since $\Gamma^t_{\ tt}=0$, we conclude that $a^t=0$.

Now consider $k=r$. The partial derivative is zero, but the second term is not, since

(0.6)
$$\Gamma^r_{ij}=\left[\begin{array}{cccc}\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right) & 0 & 0 & 0\\ 0 & -\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ 0 & 0 & -r\left(1-\frac{2GM}{r}\right) & 0\\ 0 & 0 & 0 & -r\sin^2\theta\left(1-\frac{2GM}{r}\right)\end{array}\right]$$

If $i=j=t$ we get a non-zero term:

$$du^r = u^t\Gamma^r_{tt}dx^t \tag{0.7}$$
$$= \left(1-\frac{2GM}{r}\right)^{-1/2}\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)dt \tag{0.8}$$
$$= \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{1/2}dt \tag{0.9}$$

Now we divide both sides by $d\tau$ and use the relation between S time and proper time for an object at rest:

$$\frac{dt}{d\tau}=\left(1-\frac{2GM}{r}\right)^{-1/2} \tag{0.10}$$

$$a^r=\frac{du^r}{d\tau}=\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{1/2}\left(1-\frac{2GM}{r}\right)^{-1/2}=\frac{GM}{r^2} \tag{0.11}$$

The remaining two components a^θ and a^ϕ are both zero, since $\Gamma^\theta_{tt}=\Gamma^\phi_{tt}=0$. Thus

$$\mathbf{a}=\left[0,\frac{GM}{r^2},0,0\right] \tag{0.12}$$

The magnitude can be found from the S metric:

(0.13)
$$ds^2=-\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2$$

We get

$$a = \sqrt{\mathbf{a}\cdot\mathbf{a}} \tag{0.14}$$
$$= g_{ij}a^i a^j \tag{0.15}$$
$$= \sqrt{g_{rr}}\frac{GM}{r^2} \tag{0.16}$$
$$= \frac{GM}{r^2\sqrt{1-2GM/r}} \tag{0.17}$$

This is the acceleration you would need to remain at rest at a radial distance r. Note that the acceleration becomes infinite as we approach $r = 2GM$, showing that it's impossible to remain at rest at the event horizon.