SCHWARZSCHILD METRIC: ACCELERATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.7.

An object's four-acceleration $\mathbf{a}$ is defined as

$$\mathbf{a} \equiv \frac{d\mathbf{u}}{d\tau} \tag{1}$$

For an object at rest in the Schwarzschild (S) frame, the four-velocity is given by

$$\mathbf{u} = \left[\left(1 - \frac{2GM}{r}\right)^{-1/2}, 0, 0, 0\right] \tag{2}$$

To find the acceleration, we need to take the covariant derivative of $\mathbf{u}$:

$$d\mathbf{u} = \left[\frac{\partial u^k}{\partial x^j} + u^i \Gamma^k_{\ ij}\right] \mathbf{e}_k dx^j \tag{3}$$

The individual components of $d\mathbf{u}$ are then

$$du^k = \left[\frac{\partial u^k}{\partial x^j} + u^i \Gamma^k_{\ ij}\right] dx^j \tag{4}$$

The partial derivative term is non-zero only for $k = t$ and $j = r$. Since the object is at rest, the only non-zero component of u^i is u^t, and the only non-zero differential is dx^t. Therefore, the only Christoffel symbols that we need to consider are $\Gamma^k_{\ tt}$.

The Christoffel symbols for the S metric are, for $k = t$:

$$\Gamma^t_{\ ij} = \begin{bmatrix} 0 & \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ \frac{GM}{r^2}\left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

Since $\Gamma^t_{\ tt} = 0$, we conclude that $a^t = 0$.

Now consider $k=r$. The partial derivative is zero, but the second term is not, since

(6)

$$\Gamma_{ij}^{r}=\left[\begin{array}{cccc}
\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right) & 0 & 0 & 0\\
0 & -\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\
0 & 0 & -r\left(1-\frac{2GM}{r}\right) & 0\\
0 & 0 & 0 & -r\sin^{2}\theta\left(1-\frac{2GM}{r}\right)
\end{array}\right]$$

If $i=j=t$ we get a non-zero term:

$$\begin{aligned}
du^{r} & = u^{t}\Gamma_{tt}^{r}dx^{t} && (7)\\
 & = \left(1-\frac{2GM}{r}\right)^{-1/2}\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)dt && (8)\\
 & = \frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{1/2}dt && (9)
\end{aligned}$$

Now we divide both sides by $d\tau$ and use the relation between S time and proper time for an object at rest:

$$\frac{dt}{d\tau}=\left(1-\frac{2GM}{r}\right)^{-1/2} \tag{10}$$

$$a^{r}=\frac{du^{r}}{d\tau}=\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{1/2}\left(1-\frac{2GM}{r}\right)^{-1/2}=\frac{GM}{r^{2}} \tag{11}$$

The remaining two components a^{θ} and a^{ϕ} are both zero, since $\Gamma_{tt}^{\theta}=\Gamma_{tt}^{\phi}=0$. Thus

$$\mathbf{a}=\left[0,\frac{GM}{r^{2}},0,0\right] \tag{12}$$

The magnitude can be found from the S metric:

(13)

$$ds^{2}=-\left(1-\frac{2GM}{r}\right)dt^{2}+\left(1-\frac{2GM}{r}\right)^{-1}dr^{2}+r^{2}d\theta^{2}+r^{2}\sin^{2}\theta d\phi^{2}$$

We get

$$a = \sqrt{\mathbf{a}\cdot\mathbf{a}} \tag{14}$$

$$= g_{ij}a^i a^j \tag{15}$$

$$= \sqrt{g_{rr}}\frac{GM}{r^2} \tag{16}$$

$$= \frac{GM}{r^2\sqrt{1-2GM/r}} \tag{17}$$

This is the acceleration you would need to remain at rest at a radial distance r. Note that the acceleration becomes infinite as we approach $r = 2GM$, showing that it's impossible to remain at rest at the event horizon.