

COVARIANT DERIVATIVE OF A VECTOR IN THE SCHWARZSCHILD METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.8.

Here's another example of calculating the covariant derivative in the Schwarzschild (S) metric. We're given a vector with coordinates in the S metric of:

$$(1) \quad \mathbf{v} = \left[1 - \frac{2GM}{r}, 0, 0, 0 \right]$$

The covariant derivative is given by

$$(2) \quad \nabla_j v^k \equiv \frac{\partial v^k}{\partial x^j} + v^i \Gamma_{ij}^k$$

Since the only non-zero component of \mathbf{v} is v^t and it depends only on r , most of the terms are zero.

The Christoffel symbols for the S metric are:

$$(3) \quad \Gamma_{ij}^t = \begin{bmatrix} 0 & \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(4) \quad \Gamma_{ij}^r = \begin{bmatrix} \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r \left(1 - \frac{2GM}{r}\right) & 0 \\ 0 & 0 & 0 & -r \sin^2 \theta \left(1 - \frac{2GM}{r}\right) \end{bmatrix}$$

$$(5) \quad \Gamma_{ij}^\theta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{bmatrix}$$

$$(6) \quad \Gamma_{ij}^{\phi} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot \theta \\ 0 & \frac{1}{r} & \cot \theta & 0 \end{bmatrix}$$

The one non-zero derivative is

$$(7) \quad \frac{\partial v^t}{\partial r} = \frac{2GM}{r^2}$$

and the values of the second term in 2 are

$$(8) \quad v^i \Gamma_{ir}^t = \frac{GM}{r^2}$$

$$(9) \quad v^i \Gamma_{it}^r = \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^2$$

with all other terms being zero.

The covariant derivative is then (with j the row index and k the column index):

$$(10) \quad \nabla_j v^k = \begin{bmatrix} 0 & \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^2 & 0 & 0 \\ \frac{3GM}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$