

## COVARIANT DERIVATIVE OF THE METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.9.

One interesting and useful theorem is that the covariant derivative of *any* metric tensor is always zero. We can show this by using the expression for the covariant derivative of a general tensor to say:

$$\nabla_j g_{kl} = \partial_j g_{kl} - \Gamma_{jk}^m g_{ml} - \Gamma_{jl}^m g_{km} \quad (1)$$

We can combine this with the explicit expression for the Christoffel symbols:

$$\Gamma_{ij}^m = \frac{1}{2} g^{ml} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}) \quad (2)$$

Substituting, we get

$$\nabla_j g_{kl} = \partial_j g_{kl} - \frac{1}{2} g^{mn} (\partial_k g_{jn} + \partial_j g_{nk} - \partial_n g_{kj}) g_{ml} - \frac{1}{2} g^{mn} (\partial_l g_{jn} + \partial_j g_{nl} - \partial_n g_{lj}) g_{km} \quad (3)$$

Since  $g^{mn} g_{ml} = \delta_l^n$ , we get

$$\nabla_j g_{kl} = \partial_j g_{kl} - \frac{1}{2} (\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{kj}) - \frac{1}{2} (\partial_l g_{jk} + \partial_j g_{kl} - \partial_k g_{lj}) \quad (4)$$

Now we use the symmetry of the metric tensor:  $g_{kl} = g_{lk}$ :

$$\begin{aligned} \nabla_j g_{kl} &= \partial_j g_{lk} - \frac{1}{2} (\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{kj}) - \frac{1}{2} (\partial_l g_{kj} + \partial_j g_{lk} - \partial_k g_{jl}) \quad (5) \\ &= 0 \quad (6) \end{aligned}$$

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