COVARIANT DERIVATIVE OF THE METRIC TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.9.

One interesting and useful theorem is that the covariant derivative of *any* metric tensor is always zero. We can show this by using the expression for the covariant derivative of a general tensor to say:

$$\nabla_j g_{kl} = \partial_j g_{kl} - \Gamma^m_{jk} g_{ml} - \Gamma^m_{jl} g_{km} \tag{0.1}$$

We can combine this with the explicit expression for the Christoffel symbols:

$$\Gamma^m_{ij} = \frac{1}{2} g^{ml} \left(\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji} \right) \tag{0.2}$$

Substituting, we get

$$\nabla_j g_{kl} = \partial_j g_{kl} - \frac{1}{2} g^{mn} \left(\partial_k g_{jn} + \partial_j g_{nk} - \partial_n g_{kj} \right) g_{ml} - \frac{1}{2} g^{mn} \left(\partial_l g_{jln} + \partial_j g_{nl} - \partial_n g_{lj} \right) g_{km} \tag{0.3}$$

Since $g^{mn} g_{ml} = \delta^n_l$, we get

$$\nabla_j g_{kl} = \partial_j g_{kl} - \frac{1}{2} \left(\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{kj} \right) - \frac{1}{2} \left(\partial_l g_{jk} + \partial_j g_{kl} - \partial_k g_{lj} \right) \tag{0.4}$$

Now we use the symmetry of the metric tensor: $g_{kl} = g_{lk}$:

$$\begin{aligned} \nabla_j g_{kl} &= \partial_j g_{lk} - \frac{1}{2} \left(\partial_k g_{jl} + \partial_j g_{lk} - \partial_l g_{kj} \right) - \frac{1}{2} \left(\partial_l g_{kj} + \partial_j g_{lk} - \partial_k g_{jl} \right) && (0.5) \\ &= 0 && (0.6) \end{aligned}$$

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