COVARIANT DERIVATIVE OF THE METRIC TENSOR: APPLICATION TO A COORDINATE TRANSFORMATION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.10.

Here's an application of the fact that the covariant derivative of any metric tensor is always zero. Suppose we define a coordinate transformation in which:

$$\frac{\partial x^a}{\partial x'^m} = \delta^a_m - [\Gamma^a_{\ mn}]_P \Delta x'^n_P \tag{1}$$

where $[\Gamma^a_{\ mn}]_P$ is the Christoffel symbol in the primed system evaluated at a particular point P (and therefore they are constants). (In Moore's problem P17.10, he states that this is in the unprimed system, but the problem makes no sense in that case, since we're summing over an index n which refers to the unprimed coordinate system in one term and the primed system in the other.) The quantity $\Delta x'^n_P \equiv x'^n - x'^n_p$ represents a displacement from the point P, as measured in the primed system.

Using the usual transformation equation for a tensor, we get for the metric tensor:

$$\begin{aligned}
g'_{ij} &= \frac{\partial x^a}{\partial x'^i}\frac{\partial x^b}{\partial x'^j} g_{ab} && (2)\\
&= \left(\delta^a_i - [\Gamma^a_{\ in}]_P \Delta x'^n_P\right)\left(\delta^b_j - \left[\Gamma^b_{\ jn}\right]_P \Delta x'^n_P\right) g_{ab} && (3)\\
&= g_{ij} - g_{ib}\left[\Gamma^b_{\ jn}\right]_P \Delta x'^n_P - g_{aj}\left[\Gamma^a_{\ in}\right]_P \Delta x'^n_P + [\Gamma^a_{\ is}]_P \Delta x'^s_P \left[\Gamma^b_{\ jn}\right]_P \Delta x'^n_P g_{ab} && (4)\\
&= g_{ij} - \Delta x'^n_P\left(g_{ib}\left[\Gamma^b_{\ jn}\right]_P + g_{aj}\left[\Gamma^a_{\ in}\right]_P\right) + [\Gamma^a_{\ in}]_P \left[\Gamma^b_{\ js}\right]_P \Delta x'^n_P \Delta x'^s_P g_{ab} && (5)
\end{aligned}$$

When $x' = x'_P$, $\Delta x'^n_P = 0$, so $g'_{ij} = g_{ij}$ at point P.

Now consider the second term. By renaming the dummy indices $b \to m$ and $a \to m$, we have

$$g_{ib}\left[\Gamma^b_{\ jn}\right]_P + g_{aj}\left[\Gamma^a_{\ in}\right]_P = g_{im}\left[\Gamma^m_{\ jn}\right]_P + g_{mj}\left[\Gamma^m_{\ in}\right]_P \tag{6}$$

Now because the covariant derivative (with respect to the primed coordinates) of the metric is zero, we have

$$\nabla'_n g_{ij} = \partial'_n g_{ij} - \Gamma^m_{in} g_{mj} - \Gamma^m_{jn} g_{im} = 0 \tag{7}$$

Therefore, we can write

$$g'_{ij} = g_{ij} - \partial'_n g_{ij} \Delta x'^n_P + \left[\Gamma^a_{in}\right]_P \left[\Gamma^b_{js}\right]_P \Delta x'^n_P \Delta x'^s_P g_{ab} \tag{8}$$

If we take the derivative of this with respect to a particular primed coordinate x'^k we can use

$$\frac{\partial \Delta x'^n_P}{\partial x'^k} = \delta^n_k \tag{9}$$

and then evaluate the result at $x' = x'_P$ so that all terms containing $\Delta x'^n_P$ vanish. We get

$$\begin{aligned} \frac{\partial g'_{ij}}{\partial x'^k} &= \frac{\partial g_{ij}}{\partial x'^k} - \frac{\partial g_{ij}}{\partial x'^n} \delta^n_k && (10) \\ &= 0 && (11) \end{aligned}$$

Thus all the first derivatives of g'_{ij} are zero in the primed coordinate system.