

COVARIANT DERIVATIVE OF THE METRIC TENSOR: APPLICATION TO A COORDINATE TRANSFORMATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 17; Problem P17.10.

Here's an application of the fact that the covariant derivative of any metric tensor is always zero. Suppose we define a coordinate transformation in which:

$$(1) \quad \frac{\partial x^a}{\partial x'^m} = \delta_m^a - [\Gamma_{mn}^a]_P \Delta x_P^n$$

where $[\Gamma_{mn}^a]_P$ is the Christoffel symbol in the primed system evaluated at a particular point P (and therefore they are constants). (In Moore's problem P17.10, he states that this is in the unprimed system, but the problem makes no sense in that case, since we're summing over an index n which refers to the unprimed coordinate system in one term and the primed system in the other.) The quantity $\Delta x_P^n \equiv x'^n - x_P^n$ represents a displacement from the point P , as measured in the primed system.

Using the usual transformation equation for a tensor, we get for the metric tensor:

(2)

$$g'_{ij} = \frac{\partial x^a}{\partial x'^i} \frac{\partial x^b}{\partial x'^j} g_{ab}$$

$$(3) \quad = (\delta_i^a - [\Gamma_{in}^a]_P \Delta x_P^n) \left(\delta_j^b - [\Gamma_{jn}^b]_P \Delta x_P^n \right) g_{ab}$$

$$(4) \quad = g_{ij} - g_{ib} [\Gamma_{jn}^b]_P \Delta x_P^n - g_{aj} [\Gamma_{in}^a]_P \Delta x_P^n + [\Gamma_{is}^a]_P \Delta x_P^s [\Gamma_{jn}^b]_P \Delta x_P^n g_{ab}$$

$$(5) \quad = g_{ij} - \Delta x_P^n \left(g_{ib} [\Gamma_{jn}^b]_P + g_{aj} [\Gamma_{in}^a]_P \right) + [\Gamma_{in}^a]_P [\Gamma_{js}^b]_P \Delta x_P^s \Delta x_P^n g_{ab}$$

When $x' = x'_P$, $\Delta x_P^n = 0$, so $g'_{ij} = g_{ij}$ at point P .

Now consider the second term. By renaming the dummy indices $b \rightarrow m$ and $a \rightarrow m$, we have

$$(6) \quad g_{ib} [\Gamma_{jn}^b]_P + g_{aj} [\Gamma_{in}^a]_P = g_{im} [\Gamma_{jn}^m]_P + g_{mj} [\Gamma_{in}^m]_P$$

Now because the covariant derivative (with respect to the primed coordinates) of the metric is zero, we have

$$(7) \quad \nabla'_n g_{ij} = \partial'_n g_{ij} - \Gamma^m_{in} g_{mj} - \Gamma^m_{jn} g_{im} = 0$$

Therefore, we can write

$$(8) \quad g'_{ij} = g_{ij} - \partial'_n g_{ij} \Delta x'^n_p + [\Gamma^a_{in}]_p [\Gamma^b_{js}]_p \Delta x'^n_p \Delta x'^s_p g_{ab}$$

If we take the derivative of this with respect to a particular primed coordinate x'^k we can use

$$(9) \quad \frac{\partial \Delta x'^n_p}{\partial x'^k} = \delta^n_k$$

and then evaluate the result at $x' = x'_p$ so that all terms containing $\Delta x'^n_p$ vanish. We get

$$(10) \quad \frac{\partial g'_{ij}}{\partial x'^k} = \frac{\partial g_{ij}}{\partial x'^k} - \frac{\partial g_{ij}}{\partial x'^n} \delta^n_k$$

$$(11) \quad = 0$$

Thus all the first derivatives of g'_{ij} are zero in the primed coordinate system.