NEWTONIAN TIDAL EFFECT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Box 18.1.

As a prelude to analyzing geodesic deviation in general relativity, we'll look at the tidal effect in Newtonian physics. First, though, what is geodesic deviation? A geodesic curve is a curve followed by an object in the absence of any forces (we're not considering gravity as a force here since its effects define the curvature of spacetime through which the object moves). Unless spacetime is flat, two objects initially following parallel geodesics will accelerate towards or away from each other.

This effect occurs in Newtonian physics as well, although it's not analyzed by considering spacetime to be curved. One illustration of this is to consider a box in freefall above the earth. We'll put an observer at the centre of the box, and place balls above, below and to either side of the observer. Because the gravitational force (we can think of gravity as a force if we're using Newtonian physics) due to the Earth is radial and falls off according to the inverse square formula, the ball at the top of the box experiences a force (per unit mass) less than that of the observer, who in turn feels a smaller force than the ball at the bottom of the box. Thus the ball at the top accelerates more slowly than the observer, who in turn accelerates more slowly than the ball at the bottom. To the observer, these two balls appear to be accelerating away from him in opposite directions.

Due to the radial direction of the gravitational force, the two balls on either side of the observer will feel a small component of the force pulling them towards the centre of the box (although most of the force is directed downwards, of course). As a result, the observer will see these two balls accelerating towards him. The five objects inside the box start out on parallel geodesics, but each of them follows its own geodesic, which deviates from the initial parallel path.

Qualitatively, we can see that it is this effect which causes the tides in the Earth's oceans. The Earth is in freefall about the Earth-Moon centre of mass, so points on the side of the Earth opposite the Moon have a smaller acceleration than the Earth's centre, which in turn has a smaller acceleration than points on the side closest to the Moon. Points on the surfaces of the Earth lying along the normal to the line connecting the Earth and Moon experience accelerations towards the centre of the Earth. This gives rise to high tides on the near and far points on the Earth, and low tides in between.

To make the argument quantitative, we can introduce the gravitational potential

$$\Phi = -\frac{GM}{r} \tag{1}$$

The acceleration, or force per unit mass, is the negative gradient of Φ which we can write using index notation and the flat space metric η^{ij} (which is just $\eta^{ij} = \delta^{ij}$ here):

$$\frac{d^2x^i}{dt^2} = -\eta^{ij} \left.\frac{\partial\Phi}{\partial x^j}\right|_{\vec{x}} \tag{2}$$

All quantities are evaluated at the object's position $\vec{x}(t)$. Note that everywhere in this post, the summation over indices is over only the spatial coordinates.

Now suppose we have a second object which starts off an infinitesimal separation $\vec{n}(t)$ away from the first object, so its position is $x^i(t) + n^i(t)$. The first object could be our observer in the box, and the second object could be one of the balls. To see how geodesic deviation (or the tidal effect) works, we'd like to find the relative acceleration, which is d^2n^i/dt^2.

The acceleration of the second object is

$$\frac{d^2\left(x^i + n^i\right)}{dt^2} = -\eta^{ij} \left.\frac{\partial\Phi}{\partial x^j}\right|_{\vec{x}+\vec{n}} \tag{3}$$

where this time the gradient is evaluated at $\vec{x} + \vec{n}$. If $\vec{n}$ is small, we can expand this gradient in a Taylor series about $\vec{x}$:

$$\left.\frac{\partial\Phi}{\partial x^j}\right|_{\vec{x}+\vec{n}} = \left.\frac{\partial\Phi}{\partial x^j}\right|_{\vec{x}} + n^i \left.\frac{\partial^2\Phi}{\partial x^i\partial x^j}\right|_{\vec{x}} + \ldots \tag{4}$$

Putting this into 3 we get, ignoring higher order terms:

$$\begin{aligned}
\frac{d^2\left(x^i + n^i\right)}{dt^2} &= -\eta^{ij}\left(\left.\frac{\partial\Phi}{\partial x^j}\right|_{\vec{x}} + n^k \left.\frac{\partial^2\Phi}{\partial x^k\partial x^j}\right|_{\vec{x}}\right) && (5)\\
\frac{d^2x^i}{dt^2} + \frac{d^2n^i}{dt^2} &= -\eta^{ij} \left.\frac{\partial\Phi}{\partial x^j}\right|_{\vec{x}} - \eta^{ij} n^k \left.\frac{\partial^2\Phi}{\partial x^k\partial x^j}\right|_{\vec{x}} && (6)\\
\frac{d^2n^i}{dt^2} &= -\eta^{ij} n^k \left.\frac{\partial^2\Phi}{\partial x^k\partial x^j}\right|_{\vec{x}} && (7)
\end{aligned}$$

where in the last line, we used 2. This is the Newtonian tidal deviation equation, expressed using index notation.

We can use this equation to work out the relative acceleration in the three rectangular directions for the falling box above, taking the z direction to be radially outward. Using index notation, we have

$$r = \sqrt{\eta_{mn}x^m x^n} \tag{8}$$

The gradient terms can be written as

$$\begin{aligned}
\frac{\partial\Phi}{\partial x^j} &= \frac{d\Phi}{dr}\frac{\partial r}{\partial x^j} && (9)\\
&= \frac{GM}{r^2}\frac{\eta_{jn}x^n + \eta_{mj}x^m}{2\sqrt{\eta_{mn}x^m x^n}} && (10)\\
&= \frac{GM}{r^3}\eta_{jn}x^n && (11)
\end{aligned}$$

where we got the last line using the symmetry of the metric $\eta_{ij} = \eta_{ji}$.

The second derivatives are then:

$$\begin{aligned}
\partial_k\partial_j\Phi &= -\frac{3GM}{r^4}\frac{\partial r}{\partial x^k}\eta_{jn}x^n + \frac{GM}{r^3}\eta_{jk} && (12)\\
&= -\frac{3GM}{r^4}\frac{\eta_{km}x^m}{r}\eta_{jn}x^n + \frac{GM}{r^3}\eta_{jk} && (13)\\
&= -\frac{3GM}{r^5}\eta_{km}\eta_{jn}x^n x^m + \frac{GM}{r^3}\eta_{jk} && (14)
\end{aligned}$$

Putting this into 7, and using $\eta^{ij}\eta_{jn} = \delta^i_n$:

$$\begin{aligned}
\frac{d^2n^i}{dt^2} &= -\eta^{ij}n^k\left[-\frac{3GM}{r^5}\eta_{km}\eta_{jn}x^n x^m + \frac{GM}{r^3}\eta_{jk}\right] && (15)\\
&= \frac{3GM}{r^5}\eta_{km}n^k x^m x^i - \frac{GM}{r^3}n^i && (16)
\end{aligned}$$

If we take the reference object to be at the position $x = y = 0$, $z = r$, then the first term is non-zero only when all the coordinates are z coordinates so

$$\frac{d^2n^x}{dt^2} = -\frac{GM}{r^3}n^x \tag{17}$$

$$\frac{d^2n^y}{dt^2} = -\frac{GM}{r^3}n^y \tag{18}$$

$$\frac{d^2n^z}{dt^2} = 2\frac{GM}{r^3}n^z \tag{19}$$

Thus the acceleration in the transverse directions (x and y) is opposite to the relative displacement so that the objects accelerate towards each other, and in the radial (z) direction the acceleration is away from each other.

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