TIDAL EFFECT FOR OBJECTS IN FREEFALL NEAR THE EARTH'S SURFACE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.1.

We've seen that in Newtonian physics, the tidal effect produces a relative acceleration between two objects in free fall above a sphere of mass M given by

$$\frac{d^2n^x}{dt^2} = -\frac{GM}{r^3}n^x \tag{1}$$

$$\frac{d^2n^y}{dt^2} = -\frac{GM}{r^3}n^y \tag{2}$$

$$\frac{d^2n^z}{dt^2} = 2\frac{GM}{r^3}n^z \tag{3}$$

where $\mathbf{n}$ is the relative separation vector of the two objects, and the radial direction is taken to be along the z axis.

To illustrate this, we can consider a situation where we have two masses separated initially by a vertical distance of 1 m and then released from rest near the Earth's surface. How long will it take for the distance between these objects to increase by 1 nanometre due to the tidal effect?

We'll start with the approximation that r is a constant equal to the radius of the Earth, or $r = 6.378 \times 10^6$ m. We can check the answer for consistency with this assumption after we're done. We'll also need the mass of the Earth: $M = 5.98 \times 10^{24}$ kg and of course $G = 6.67 \times 10^{-11}$ in MKS units. We're thus faced with the differential equation:

$$\ddot{n}^z = 3.075 \times 10^{-6} n^z \tag{4}$$

This has the general solution

$$n^z(t) = Ae^{0.00175t} + Be^{-0.00175t} \tag{5}$$

where the numerical factor in the exponent is $\sqrt{3.075 \times 10^{-6}}$. From initial conditions

$$
\begin{aligned}
n^z(0) &= 1 = A + B && (6)\\
\dot{n}^z(0) &= 0 = 0.00175\,(A - B) && (7)
\end{aligned}
$$

Therefore

$$A = B = \frac{1}{2} \tag{8}$$

We want the time t such that $n^z(t) = 1 + 10^{-9}$ m or in other words:

$$
\begin{aligned}
\frac{1}{2}\left(e^{0.00175t} + e^{-0.00175t}\right) - 1 &= 10^{-9} && (9)\\
\cosh 0.00175t - 1 &= 10^{-9} && (10)
\end{aligned}
$$

Since the difference is so small, we can expand the argument of the cosh in a Taylor series and keep the leading term:

$$
\begin{aligned}
\cosh 0.00175t - 1 &= \frac{1}{2}(0.00175t)^2 + \mathcal{O}\left(t^4\right) && (11)\\
&= 10^{-9} && (12)\\
t &= 0.0255 \text{ s} && (13)
\end{aligned}
$$

In that time, the objects would fall a distance of

$$d = \frac{1}{2}gt^2 = 0.003 \text{ m}$$

or about 3 mm, so the assumption of r being constant at the Earth's radius seems reasonable.

For the separation to reach 1 mm, the same calculation yields a time of around 25.5 sec, which would amount to a distance of (assuming the acceleration due to gravity g is constant and ignoring air resistance) of around 3.2 km, so here the assumption of a constant r is a bit shakier, considering we're dealing with such a small tidal effect.