RIEMANN TENSOR: DERIVATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Boxes 18.2, 18.3, 18.4.

We can use the Newtonian calculation of the tidal effect to derive the analogous condition in general relativity. In the Newtonian case, the objects in the falling box follow paths given by Newton's laws of motion in a gravitational field. In GR, the presence of mass causes spacetime to curve, and freely falling objects follow geodesics determined by that curvature. The geodesic equation in terms of Christoffel symbols is

$$\ddot{x}^m+\Gamma_{ij}^m\dot{x}^j\dot{x}^i=0 \tag{1}$$

where the dot indicates a derivative with respect to proper time τ.

The idea is to explore this equation for two objects that momentarily follow geodesics that are parallel and separated by an infinitesimal distance, which we call n^i. That is, we write the path followed by our reference object as $x^k(\tau)$, where τ is that object's proper time, and by the other object as $\overline{x}^i(\tau)$. The separation is thus

$$n^i(\tau)\equiv\overline{x}^i(\tau)-x^k(\tau) \tag{2}$$

Note that it is the *same* proper time (that is, the proper time of the reference object) that is used in all three terms. Although both x^i and $\overline{x}^i$ are four-vectors and we can in principle use any quantity such as τ to parametrize them, the *difference* of these two vectors where we use the proper time of one object to parametrize both vectors will not, in general, be a four-vector. It would be a four-vector only if the two geodesics are parallel, in which case the proper times of both objects would be the same.

However, for the purposes of the argument here, we are considering only a little chunk of spacetime in which the two geodesics happen to be separated by the infinitesimal distance n^i and in that location the two geodesics are assumed to be virtually parallel, so the proper time τ *can* be used as the proper time of both objects. For that small section of spacetime, n^i can be taken as a four-vector. Under these conditions, the geodesic equations for the two objects are:

$$\ddot{x}^m + \Gamma^m_{ij}\dot{x}^j\dot{x}^i = 0 \tag{3}$$

$$\ddot{\overline{x}}^m + \overline{\Gamma}^m_{ij}\dot{\overline{x}}^j\dot{\overline{x}}^i = 0 \tag{4}$$

The conditions of these equations allow us to take the derivatives in both cases with respect to the same proper time τ. The Christoffel symbol in the second equation is evaluated at $\overline{x}^i$, that is

$$\overline{\Gamma}^m_{ij} = \Gamma^m_{ij}\left(\overline{x}^k(\tau)\right) \tag{5}$$

Now since the barred coordinates are separated from the reference coordinates by an infinitesimal amount, we can get the barred Christoffel symbols from the unbarred ones by using a Taylor expansion. That is

$$\overline{\Gamma}^m_{ij} = \Gamma^m_{ij}\left(x^k(\tau) + n^k(\tau)\right) \tag{6}$$

$$= \Gamma^m_{ij}\left(x^k(\tau)\right) + n^\ell\partial_\ell\Gamma^m_{ij}\left(x^k(\tau)\right) + \ldots \tag{7}$$

By putting 2 and 7 into 4 we can eliminate all barred terms:

$$\ddot{x}^m + \ddot{n}^m + \left(\Gamma^m_{ij} + n^\ell\partial_\ell\Gamma^m_{ij}\right)\left(\dot{x}^j + \dot{n}^j\right)\left(\dot{x}^i + \dot{n}^i\right) = 0 \tag{8}$$

Multiplying this out and ignoring all terms of second order or higher in n and its derivatives, we find

$$\ddot{x}^m + \Gamma^m_{ij}\dot{x}^j\dot{x}^i + \ddot{n}^m + \Gamma^m_{ij}\left(\dot{x}^j\dot{n}^i + \dot{n}^j\dot{x}^i\right) + \dot{x}^i\dot{x}^j n^\ell\partial_\ell\Gamma^m_{ij} = 0 \tag{9}$$

We can use 3 to eliminate the first two terms and the symmetry of $\Gamma^m_{ij} = \Gamma^m_{ji}$ to simplify the third term to get

$$\ddot{n}^m + 2\Gamma^m_{ij}\dot{x}^j\dot{n}^i + \dot{x}^i\dot{x}^j n^\ell\partial_\ell\Gamma^m_{ij} = 0 \tag{10}$$

Using the four-velocity $u^i \equiv \dot{x}^i$ we can write this as

$$\ddot{n}^m + 2\Gamma^m_{ij}u^j\dot{n}^i + u^iu^jn^\ell\partial_\ell\Gamma^m_{ij} = 0 \tag{11}$$

We now have an expression for $\ddot{n}^m$, but as usual, this isn't the total derivative of the four-vector $\mathbf{n}$, since its derivative could also get a contribution from the change of the basis vectors $\mathbf{e}_i$ as the object moves along its geodesic. To get the total derivative, we have

$$\dot{\mathbf{n}} = \frac{d}{d\tau}\left(n^i \mathbf{e}_i\right) = \dot{n}^i \mathbf{e}_i + n^i \dot{\mathbf{e}}_i \tag{12}$$

From the definition of the Christoffel symbols, we have

$$\frac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma^k_{\ ij} \mathbf{e}_k \tag{13}$$

so we can use the chain rule to write

$$\dot{\mathbf{e}}_i = \dot{x}^j \partial_j \mathbf{e}_i = u^j \Gamma^k_{\ ij} \mathbf{e}_k \tag{14}$$

so

$$\dot{\mathbf{n}} = \dot{n}^i \mathbf{e}_i + n^i u^j \Gamma^k_{\ ij} \mathbf{e}_k \tag{15}$$

$$= \dot{n}^i \mathbf{e}_i + n^k u^j \Gamma^i_{\ kj} \mathbf{e}_i \tag{16}$$

$$\dot{\mathbf{n}}^i = \dot{n}^i + n^k u^j \Gamma^i_{\ kj} \tag{17}$$

The result 17 applies to any four-vector $\mathbf{n}$. Since we're still dealing with the condition that $\mathbf{n}$ is a four-vector, its derivative with respect to proper time is also a four-vector, so we can find the *second* absolute derivative by using 17 on $\dot{\mathbf{n}}$:

$$\ddot{\mathbf{n}}^i = \frac{d\dot{\mathbf{n}}^i}{d\tau} + \dot{\mathbf{n}}^k u^j \Gamma^i_{\ kj} \tag{18}$$

$$= \frac{d}{d\tau}\left[\dot{n}^i + n^k u^j \Gamma^i_{\ kj}\right] + \left[\dot{n}^k + n^\ell u^j \Gamma^k_{\ \ell j}\right] u^m \Gamma^i_{\ km} \tag{19}$$

$$= \ddot{n}^i + \dot{n}^k u^j \Gamma^i_{\ kj} + n^k \dot{u}^j \Gamma^i_{\ kj} + n^k u^j \dot{\Gamma}^i_{\ kj} + \dot{n}^k u^m \Gamma^i_{\ km} + n^\ell u^j u^m \Gamma^k_{\ \ell j} \Gamma^i_{\ km} \tag{20}$$

We can use the chain rule on the third term:

$$\dot{\Gamma}^i_{\ kj} = \dot{x}^\ell \partial_\ell \Gamma^i_{\ kj} = u^\ell \partial_\ell \Gamma^i_{\ kj}$$

Substituting this and 11 into 20 we get

$$\begin{aligned}\ddot{\mathbf{n}}^i = &-2\Gamma^i_{\ \ell j} u^j \dot{n}^\ell - u^m u^j n^\ell \partial_\ell \Gamma^i_{\ mj} + \dot{n}^k u^j \Gamma^i_{\ kj} + n^k \dot{u}^j \Gamma^i_{\ kj} \\ &+ n^k u^j u^\ell \partial_\ell \Gamma^i_{\ kj} + \dot{n}^k u^m \Gamma^i_{\ km} + n^\ell u^j u^m \Gamma^k_{\ \ell j} \Gamma^i_{\ km}\end{aligned} \tag{21}$$

By relabelling indices, we see that the first term cancels the third and sixth terms, so we have

$$\ddot{\mathbf{n}}^{i}=-u^{m}u^{j}n^{\ell}\partial_{\ell}\Gamma_{\ mj}^{i}+n^{k}\dot{u}^{j}\Gamma_{\ kj}^{i}+n^{k}u^{j}u^{\ell}\partial_{\ell}\Gamma_{\ kj}^{i}+n^{\ell}u^{j}u^{m}\Gamma_{\ \ell j}^{k}\Gamma_{\ km}^{i} \tag{22}$$

We can write 3 in terms of the four-velocity:

$$\dot{u}^{j}=-\Gamma_{\ m\ell}^{j}u^{\ell}u^{m} \tag{23}$$

and insert this into the second term to get

$$\ddot{\mathbf{n}}^{i}=-u^{m}u^{j}n^{\ell}\partial_{\ell}\Gamma_{\ mj}^{i}-n^{k}u^{\ell}u^{m}\Gamma_{\ m\ell}^{j}\Gamma_{\ kj}^{i}+n^{k}u^{j}u^{\ell}\partial_{\ell}\Gamma_{\ kj}^{i}+n^{\ell}u^{j}u^{m}\Gamma_{\ \ell j}^{k}\Gamma_{\ km}^{i} \tag{24}$$

Now we need to relabel indices so that we can factor out $u^{m}u^{j}n^{\ell}$. We can do this by the following switches:

Term 2:	$j\to k$; $\ell\to j$; $k\to\ell$
Term 3:	$\ell\to m$; $k\to\ell$
Term 4:	No changes needed.

We get:

$$\ddot{\mathbf{n}}^{i}=u^{m}u^{j}n^{\ell}\left[\partial_{m}\Gamma_{\ \ell j}^{i}-\partial_{\ell}\Gamma_{\ mj}^{i}+\Gamma_{\ \ell j}^{k}\Gamma_{\ km}^{i}-\Gamma_{\ mj}^{k}\Gamma_{\ \ell k}^{i}\right] \tag{25}$$

Since this is still a tensor equation, the quantity in brackets is a tensor and is called the *Riemann tensor*.

$$\boxed{R_{\ j\ell m}^{i}\equiv\pm\left[\partial_{m}\Gamma_{\ \ell j}^{i}-\partial_{\ell}\Gamma_{\ mj}^{i}+\Gamma_{\ \ell j}^{k}\Gamma_{\ km}^{i}-\Gamma_{\ mj}^{k}\Gamma_{\ \ell k}^{i}\right]} \tag{26}$$

We've written a $\pm$ in the definition, since some books use one sign while others use the other one. Moore uses the minus sign, so we'll stick with that in these posts. We now have a compact expression for geodesic deviation:

$$\boxed{\ddot{\mathbf{n}}^{i}=-R_{\ j\ell m}^{i}u^{m}u^{j}n^{\ell}} \tag{27}$$

This is the equation of geodesic deviation in general relativity. Its interpretation is that if the relative acceleration $\ddot{\mathbf{n}}^{i}=0$ then spacetime is flat; otherwise it's curved. Since the only quantity in this equation that depends intrinsically on the metric is $R_{\ \ell mj}^{i}$, we see that if the Riemann tensor is identically zero, spacetime is flat, but if only one component of this tensor is non-zero, spacetime is curved. We'll deal with some more properties and examples of the Riemann tensor in future posts.

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