RIEMANN TENSOR FOR AN INFINITE PLANE OF MASS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.3.

We've seen that an infinite plane of mass causes no tidal effects in Newtonian physics. If this were also true in general relativity, that is, if there were no geodesic deviation, then according to GR, space is flat.

If we take the x axis perpendicular to the plane, then the system is completely symmetric in the t, y and z dimensions, so any metric tensor describing spacetime in this system can depend only on x. If take the metric tensor to be diagonal, then we get

$$ds^2 = -dt^2 + f(x)\,dx^2 + dy^2 + dz^2 \tag{1}$$

$$g_{ij} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & f(x) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where $f(x)$ is an arbitrary function of x only. If the Riemann tensor for this metric is identically zero, then we can say that this metric describes a flat spacetime. The Riemann tensor is

$$R^i_{\ j\ell m} \equiv \pm\left[\partial_m \Gamma^i_{\ \ell j} - \partial_\ell \Gamma^i_{\ mj} + \Gamma^k_{\ \ell j}\Gamma^i_{\ km} - \Gamma^k_{\ mj}\Gamma^i_{\ \ell k}\right] \tag{3}$$

so we need the Christoffel symbols, which we can get by comparing the two forms of the geodesic equation. These equations are

$$g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j\dot{x}^i = 0 \tag{4}$$

$$\ddot{x}^m + \Gamma^m_{ij}\dot{x}^j\dot{x}^i = 0 \tag{5}$$

Using the first equation, we see that if $a = t$, y or z, the geodesic equation is

$$g_{aa}\ddot{x}^a = 0 \text{ (no sum)} \tag{6}$$

For $a = x$ we get

$$f(x)\ddot{x}+\frac{1}{2}f'(x)\dot{x}^2=0 \tag{7}$$

Dividing through by $f(x)$ and comparing with 5, we see that the only non-zero Christoffel symbol is

$$\Gamma_{xx}^{x}=\frac{f'(x)}{2f(x)} \tag{8}$$

From 3, the only possibly non-zero component of the Riemann tensor is R_{xxx}^{x}, since the only non-zero Christoffel symbol is Γ_{xx}^{x} and the only non-zero derivative of the Christoffel symbols is with respect to x. However, in this case the first two terms in 3 cancel, as do the last two, so $R^{i}{}_{\ell mj}=0$ for all components. Thus spacetime is indeed flat for this metric.