RIEMANN TENSOR IN 2-D POLAR COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.4.

As a simple example of the Riemann tensor we can work out one of its components in 2-d polar coordinates. The tensor is

$$R^i{}_{j\ell m} \equiv \partial_\ell \Gamma^i{}_{mj} - \partial_m \Gamma^i{}_{\ell j} + \Gamma^k{}_{mj} \Gamma^i{}_{\ell k} - \Gamma^k{}_{\ell j} \Gamma^i{}_{km} \tag{1}$$

As usual, we need the Christoffel symbols, which we can get by comparing the two forms of the geodesic equation. These equations are

$$\begin{aligned} g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j \dot{x}^i &= 0 && (2)\\ \ddot{x}^m + \Gamma^m_{ij}\dot{x}^j \dot{x}^i &= 0 && (3) \end{aligned}$$

The metric is

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{4}$$

so $g_{rr} = 1$ and $g_{\theta\theta} = r^2$. For the two coordinates, 2 gives us

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= 0 && (5)\\ r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} &= 0 && (6) \end{aligned}$$

Comparing with 3 we get

$$\begin{aligned} \Gamma^r{}_{\theta\theta} &= -r && (7)\\ \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} &= \frac{1}{r} && (8) \end{aligned}$$

One component of the Riemann tensor is :

$$\begin{aligned} R^r{}_{\theta r\theta} &= \partial_r \Gamma^r{}_{\theta\theta} - \partial_\theta \Gamma^r{}_{r\theta} + \Gamma^r{}_{kr}\Gamma^k{}_{\theta\theta} - \Gamma^r{}_{\theta k}\Gamma^k{}_{r\theta} && (9)\\ &= -1 - 0 + 0 + 1 && (10)\\ &= 0 && (11) \end{aligned}$$

In fact, there is only one independent component of R in a 2-d metric (as we'll see when we study the symmetry of R later), so all other components can be written as multiples of $R^{r}{}_{\theta r\theta}$. Thus for polar coordinates $R^{r}{}_{\theta r\theta} = 0$ implies all components of R are zero, which is what we'd expect since polar coordinates describe flat space.