RIEMANN TENSOR IN 2-D CURVED SPACE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.5.

Here's another example of the Riemann tensor in a 2-d coordinate system. The tensor is

$$R^{i}{}_{j\ell m}\equiv\partial_{\ell}\Gamma^{i}{}_{mj}-\partial_{m}\Gamma^{i}{}_{\ell j}+\Gamma^{k}{}_{mj}\Gamma^{i}{}_{\ell k}-\Gamma^{k}{}_{\ell j}\Gamma^{i}{}_{km} \tag{1}$$

As usual, we need the Christoffel symbols, which we can get by comparing the two forms of the geodesic equation. These equations are

$$g_{aj}\ddot{x}^{j}+\left(\partial_{i}g_{aj}-\frac{1}{2}\partial_{a}g_{ij}\right)\dot{x}^{j}\dot{x}^{i} = 0 \tag{2}$$

$$\ddot{x}^{m}+\Gamma_{ij}^{m}\dot{x}^{j}\dot{x}^{i} = 0 \tag{3}$$

The metric is

$$ds^{2}=\frac{dp^{2}}{1-kp^{2}}+p^{2}dq^{2} \tag{4}$$

so $g_{pp}=1/\left(1-kp^{2}\right)$ and $g_{qq}=p^{2}$. For the two coordinates, 2 gives us

$$\frac{1}{1-kp^{2}}\ddot{p}+\frac{kp}{\left(1-kp^{2}\right)^{2}}\dot{p}^{2}-p\dot{q}^{2} = 0 \tag{5}$$

$$p^{2}\ddot{q}+2p\dot{p}\dot{q} = 0 \tag{6}$$

Dividing through by the coefficient of the second derivative in each case gives:

$$\ddot{p}+\frac{kp}{1-kp^{2}}\dot{p}^{2}-p\left(1-kp^{2}\right)\dot{q} = 0 \tag{7}$$

$$\ddot{q}+\frac{2}{p}\dot{p}\dot{q} = 0 \tag{8}$$

Comparing with 3 we get

$$\Gamma^{p}_{pp} = \frac{kp}{1-kp^2} \tag{9}$$
$$\Gamma^{p}_{qq} = -p\left(1-kp^2\right) \tag{10}$$
$$\Gamma^{q}_{pq} = \Gamma^{q}_{qp} = \frac{1}{p} \tag{11}$$

The only independent Riemann tensor component in 2-d is R^{p}_{qpq} :

$$R^{p}_{qpq} = \partial_p \Gamma^{p}_{qq} - \partial_q \Gamma^{p}_{pq} + \Gamma^{p}_{kp}\Gamma^{k}_{qq} - \Gamma^{p}_{qk}\Gamma^{k}_{pq} \tag{12}$$
$$= -\left(1-3kp^2\right) - 0 - kp^2 + \left(1-kp^2\right) \tag{13}$$
$$= kp^2 \tag{14}$$

Any non-zero component indicates that the space is curved, so this metric represents a curved space.