

RIEMANN TENSOR IN 2-D CURVED SPACE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.5.

Here's another example of the Riemann tensor in a 2-d coordinate system. The tensor is

$$R^i{}_{j\ell m} \equiv \partial_\ell \Gamma^i{}_{mj} - \partial_m \Gamma^i{}_{\ell j} + \Gamma^k{}_{mj} \Gamma^i{}_{\ell k} - \Gamma^k{}_{\ell j} \Gamma^i{}_{km} \quad (1)$$

As usual, we need the Christoffel symbols, which we can get by comparing the two forms of the geodesic equation. These equations are

$$g_{aj} \ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2} \partial_a g_{ij} \right) \dot{x}^j \dot{x}^i = 0 \quad (2)$$

$$\ddot{x}^m + \Gamma^m{}_{ij} \dot{x}^j \dot{x}^i = 0 \quad (3)$$

The metric is

$$ds^2 = \frac{dp^2}{1 - kp^2} + p^2 dq^2 \quad (4)$$

so $g_{pp} = 1/(1 - kp^2)$ and $g_{qq} = p^2$. For the two coordinates, 2 gives us

$$\frac{1}{1 - kp^2} \ddot{p} + \frac{kp}{(1 - kp^2)^2} \dot{p}^2 - p \dot{q}^2 = 0 \quad (5)$$

$$p^2 \ddot{q} + 2p \dot{p} \dot{q} = 0 \quad (6)$$

Dividing through by the coefficient of the second derivative in each case gives:

$$\ddot{p} + \frac{kp}{1 - kp^2} \dot{p}^2 - p(1 - kp^2) \dot{q} = 0 \quad (7)$$

$$\ddot{q} + \frac{2}{p} \dot{p} \dot{q} = 0 \quad (8)$$

Comparing with 3 we get

$$\Gamma_{pp}^p = \frac{kp}{1 - kp^2} \quad (9)$$

$$\Gamma_{qq}^p = -p(1 - kp^2) \quad (10)$$

$$\Gamma_{pq}^q = \Gamma_{qp}^q = \frac{1}{p} \quad (11)$$

The only independent Riemann tensor component in 2-d is R_{ppq}^p :

$$R_{ppq}^p = \partial_p \Gamma_{qq}^p - \partial_q \Gamma_{pq}^p + \Gamma_{kp}^p \Gamma_{qq}^k - \Gamma_{qk}^p \Gamma_{pq}^k \quad (12)$$

$$= -(1 - 3kp^2) - 0 - kp^2 + (1 - kp^2) \quad (13)$$

$$= kp^2 \quad (14)$$

Any non-zero component indicates that the space is curved, so this metric represents a curved space.