RIEMANN TENSOR IN THE SCHWARZSCHILD METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.6.

We'll calculate one component of the Riemann tensor for the Schwarzschild metric. The tensor is

$$R^{i}{}_{j\ell m}\equiv\partial_{\ell}\Gamma^{i}{}_{mj}-\partial_{m}\Gamma^{i}{}_{\ell j}+\Gamma^{k}{}_{mj}\Gamma^{i}{}_{\ell k}-\Gamma^{k}{}_{\ell j}\Gamma^{i}{}_{km} \tag{0.1}$$

As usual, we need the Christoffel symbols, but we've already worked these out.

$$\Gamma^{t}{}_{ij}=\left[\begin{array}{cccc}0 & \frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ \frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \tag{0.2}$$

$$\Gamma^{r}{}_{ij}=\left[\begin{array}{cccc}\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right) & 0 & 0 & 0\\ 0 & -\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ 0 & 0 & -r\left(1-\frac{2GM}{r}\right) & 0\\ 0 & 0 & 0 & -r\sin^{2}\theta\left(1-\frac{2GM}{r}\right)\end{array}\right] \tag{0.3}$$

$$\Gamma^{\theta}{}_{ij}=\left[\begin{array}{cccc}0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{r} & 0\\ 0 & \frac{1}{r} & 0 & 0\\ 0 & 0 & 0 & -\sin\theta\cos\theta\end{array}\right] \tag{0.4}$$

$$\Gamma^{\phi}{}_{ij}=\left[\begin{array}{cccc}0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{r}\\ 0 & 0 & 0 & \cot\theta\\ 0 & \frac{1}{r} & \cot\theta & 0\end{array}\right] \tag{0.5}$$

We can plug these into the formula above to get $R^{t}{}_{rtr}$. We have

$$R^t{}_{rtr} = \partial_t \Gamma^t{}_{rr} - \partial_r \Gamma^t{}_{tr} + \Gamma^k{}_{rr} \Gamma^t{}_{kt} - \Gamma^k{}_{tr} \Gamma^t{}_{rk} \tag{0.6}$$

We can work out these terms one at a time (only the index k is summmed):

$$\partial_t \Gamma^t{}_{rr} = 0 \tag{0.7}$$

$$-\partial_r \Gamma^t{}_{tr} = \frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1} + \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-2}\left(\frac{2GM}{r^2}\right) \tag{0.8}$$

$$= \frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1} + \frac{2G^2M^2}{r^4}\left(1-\frac{2GM}{r}\right)^{-2} \tag{0.9}$$

$$\Gamma^k{}_{rr}\Gamma^t{}_{kt} = \Gamma^r{}_{rr}\Gamma^t{}_{rt} \tag{0.10}$$

$$= -\frac{G^2M^2}{r^4}\left(1-\frac{2GM}{r}\right)^{-2} \tag{0.11}$$

(0.12)

$$-\Gamma^k{}_{tr}\Gamma^t{}_{rk} = -\left(\Gamma^t{}_{rt}\right)^2$$

$$= -\frac{G^2M^2}{r^4}\left(1-\frac{2GM}{r}\right)^{-2} \tag{0.13}$$

Adding these up we get

$$R^t{}_{rtr} = \frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1} \tag{0.14}$$

Since this is never zero, Schwarzschild spacetime is curved everywhere, but as $r \to \infty$, $R^t{}_{rtr} \to 0$ so the further we get from the mass, the less curved the spacetime becomes.

PINGBACKS

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