RIEMANN TENSOR IN THE SCHWARZSCHILD METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.6.

We'll calculate one component of the Riemann tensor for the Schwarzschild metric. The tensor is

$$R^{i}{}_{j\ell m} \equiv \partial_\ell \Gamma^{i}{}_{mj} - \partial_m \Gamma^{i}{}_{\ell j} + \Gamma^{k}{}_{mj}\Gamma^{i}{}_{\ell k} - \Gamma^{k}{}_{\ell j}\Gamma^{i}{}_{km} \tag{1}$$

As usual, we need the Christoffel symbols, but we've already worked these out.

$$\Gamma^{t}{}_{ij} = \begin{bmatrix} 0 & \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 \\ \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2}$$

$$\Gamma^{r}{}_{ij} = \begin{bmatrix} \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r\left(1-\frac{2GM}{r}\right) & 0 \\ 0 & 0 & 0 & -r\sin^2\theta\left(1-\frac{2GM}{r}\right) \end{bmatrix} \tag{3}$$

$$\Gamma^{\theta}{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix} \tag{4}$$

$$\Gamma^{\phi}{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{bmatrix} \tag{5}$$

We can plug these into the formula above to get $R^{t}{}_{rtr}$. We have

$$R^{t}{}_{rtr} = \partial_t \Gamma^{t}{}_{rr} - \partial_r \Gamma^{t}{}_{tr} + \Gamma^{k}{}_{rr}\Gamma^{t}{}_{kt} - \Gamma^{k}{}_{tr}\Gamma^{t}{}_{rk} \tag{6}$$

We can work out these terms one at a time (only the index k is summmed):

$$\partial_t \Gamma^t_{\ rr} = 0 \tag{7}$$

$$-\partial_r \Gamma^t_{\ tr} = \frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1} + \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-2}\left(\frac{2GM}{r^2}\right) \tag{8}$$

$$= \frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1} + \frac{2G^2M^2}{r^4}\left(1-\frac{2GM}{r}\right)^{-2} \tag{9}$$

$$\Gamma^k_{\ rr}\Gamma^t_{\ kt} = \Gamma^r_{\ rr}\Gamma^t_{\ rt} \tag{10}$$

$$= -\frac{G^2M^2}{r^4}\left(1-\frac{2GM}{r}\right)^{-2} \tag{11}$$

$$-\Gamma^k_{\ tr}\Gamma^t_{\ rk} = -\left(\Gamma^t_{\ rt}\right)^2 \tag{12}$$

$$= -\frac{G^2M^2}{r^4}\left(1-\frac{2GM}{r}\right)^{-2} \tag{13}$$

Adding these up we get

$$R^t_{\ rtr} = \frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1} \tag{14}$$

Since this is never zero, Schwarzschild spacetime is curved everywhere, but as $r \to \infty$, $R^t_{\ rtr} \to 0$ so the further we get from the mass, the less curved the spacetime becomes.

PINGBACKS

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