RIEMANN TENSOR IN THE SCHWARZSCHILD METRIC

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.6.

We'll calculate one component of the Riemann tensor for the Schwarzschild metric. The tensor is

$$R^{i}_{\ j\ell m}\equiv\partial_{\ell}\Gamma^{i}_{\ mj}-\partial_{m}\Gamma^{i}_{\ \ell j}+\Gamma^{k}_{\ mj}\Gamma^{i}_{\ \ell k}-\Gamma^{k}_{\ \ell j}\Gamma^{i}_{\ km} \tag{1}$$

As usual, we need the Christoffel symbols, but we've already worked these out.

$$\Gamma^{t}_{\ ij}=\left[\begin{array}{cccc}0 & \frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ \frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right] \tag{2}$$

$$\Gamma^{r}_{\ ij}=\left[\begin{array}{cccc}\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right) & 0 & 0 & 0\\ 0 & -\frac{GM}{r^{2}}\left(1-\frac{2GM}{r}\right)^{-1} & 0 & 0\\ 0 & 0 & -r\left(1-\frac{2GM}{r}\right) & 0\\ 0 & 0 & 0 & -r\sin^{2}\theta\left(1-\frac{2GM}{r}\right)\end{array}\right] \tag{3}$$

$$\Gamma^{\theta}_{\ ij}=\left[\begin{array}{cccc}0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{r} & 0\\ 0 & \frac{1}{r} & 0 & 0\\ 0 & 0 & 0 & -\sin\theta\cos\theta\end{array}\right] \tag{4}$$

$$\Gamma^{\phi}_{\ ij}=\left[\begin{array}{cccc}0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{r}\\ 0 & 0 & 0 & \cot\theta\\ 0 & \frac{1}{r} & \cot\theta & 0\end{array}\right] \tag{5}$$

We can plug these into the formula above to get $R^{t}_{\ rtr}$. We have

$$R^{t}_{\ rtr}=\partial_{t}\Gamma^{t}_{\ rr}-\partial_{r}\Gamma^{t}_{\ tr}+\Gamma^{k}_{\ rr}\Gamma^{t}_{\ kt}-\Gamma^{k}_{\ tr}\Gamma^{t}_{\ rk} \tag{6}$$

We can work out these terms one at a time (only the index k is summed):

$$\partial_t \Gamma^t_{\ rr} = 0 \tag{7}$$

$$-\partial_r \Gamma^t_{\ tr} = \frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1} + \frac{GM}{r^2}\left(1-\frac{2GM}{r}\right)^{-2}\left(\frac{2GM}{r^2}\right) \tag{8}$$

$$= \frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1} + \frac{2G^2M^2}{r^4}\left(1-\frac{2GM}{r}\right)^{-2} \tag{9}$$

$$\Gamma^k_{\ rr}\Gamma^t_{\ kt} = \Gamma^r_{\ rr}\Gamma^t_{\ rt} \tag{10}$$

$$= -\frac{G^2M^2}{r^4}\left(1-\frac{2GM}{r}\right)^{-2} \tag{11}$$

$$-\Gamma^k_{\ tr}\Gamma^t_{\ rk} = -\left(\Gamma^t_{\ rt}\right)^2 \tag{12}$$

$$= -\frac{G^2M^2}{r^4}\left(1-\frac{2GM}{r}\right)^{-2} \tag{13}$$

Adding these up we get

$$R^t_{\ rtr} = \frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1} \tag{14}$$

Since this is never zero, Schwarzschild spacetime is curved everywhere, but as $r \to \infty$, $R^t_{\ rtr} \to 0$ so the further we get from the mass, the less curved the spacetime becomes.

PINGBACKS

Pingback: Riemann tensor in the Schwarzschild metric: observer's view