

RIEMANN TENSOR IN THE SCHWARZSCHILD METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.6.

We'll calculate one component of the Riemann tensor for the Schwarzschild metric. The tensor is

$$(1) \quad R^i{}_{j\ell m} \equiv \partial_\ell \Gamma^i{}_{mj} - \partial_m \Gamma^i{}_{\ell j} + \Gamma^k{}_{mj} \Gamma^i{}_{\ell k} - \Gamma^k{}_{\ell j} \Gamma^i{}_{km}$$

As usual, we need the Christoffel symbols, but we've already worked these out.

(2)

$$\Gamma^t{}_{ij} = \begin{bmatrix} 0 & \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(3)

$$\Gamma^r{}_{ij} = \begin{bmatrix} \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right) & 0 & 0 & 0 \\ 0 & -\frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r \left(1 - \frac{2GM}{r}\right) & 0 \\ 0 & 0 & 0 & -r \sin^2 \theta \left(1 - \frac{2GM}{r}\right) \end{bmatrix}$$

(4)

$$\Gamma^\theta{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin \theta \cos \theta \end{bmatrix}$$

(5)

$$\Gamma^\phi{}_{ij} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot \theta \\ 0 & \frac{1}{r} & \cot \theta & 0 \end{bmatrix}$$

We can plug these into the formula above to get $R^t{}_{rtr}$. We have

$$(6) \quad R^t_{rr} = \partial_t \Gamma^t_{rr} - \partial_r \Gamma^t_{tr} + \Gamma^k_{rr} \Gamma^t_{kt} - \Gamma^k_{tr} \Gamma^t_{rk}$$

We can work out these terms one at a time (only the index k is summed):

$$(7) \quad \partial_t \Gamma^t_{rr} = 0$$

$$(8) \quad -\partial_r \Gamma^t_{tr} = \frac{2GM}{r^3} \left(1 - \frac{2GM}{r}\right)^{-1} + \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-2} \left(\frac{2GM}{r^2}\right)$$

$$(9) \quad = \frac{2GM}{r^3} \left(1 - \frac{2GM}{r}\right)^{-1} + \frac{2G^2M^2}{r^4} \left(1 - \frac{2GM}{r}\right)^{-2}$$

$$(10) \quad \Gamma^k_{rr} \Gamma^t_{kt} = \Gamma^r_{rr} \Gamma^t_{rt}$$

$$(11) \quad = -\frac{G^2M^2}{r^4} \left(1 - \frac{2GM}{r}\right)^{-2}$$

$$(12) \quad -\Gamma^k_{tr} \Gamma^t_{rk} = -(\Gamma^t_{rt})^2$$

$$(13) \quad = -\frac{G^2M^2}{r^4} \left(1 - \frac{2GM}{r}\right)^{-2}$$

Adding these up we get

$$(14) \quad R^t_{rr} = \frac{2GM}{r^3} \left(1 - \frac{2GM}{r}\right)^{-1}$$

Since this is never zero, Schwarzschild spacetime is curved everywhere, but as $r \rightarrow \infty$, $R^t_{rr} \rightarrow 0$ so the further we get from the mass, the less curved the spacetime becomes.

PINGBACKS

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