GEODESIC DEVIATION IN A LOCALLY INERTIAL FRAME

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.7.

Consider a coordinate frame attached to an object that is freely falling (that is, it's following a geodesic). We can choose the coordinates such that they satisfy the conditions for a locally inertial frame (LIF), and they remain so as long as the object remains in free fall. This means that the Christoffel symbols are zero, since they are defined in terms of the metric tensor by

$$\Gamma_{ij}^{m}=\frac{1}{2}g^{ml}\left(\partial_{j}g_{il}+\partial_{i}g_{lj}-\partial_{l}g_{ji}\right) \tag{1}$$

and in a LIF all first derivatives of g_{ij} are zero.

An object at rest in a LIF has a four velocity of $u^{i}=[1,0,0,0]$ in that frame, so the equation of geodesic deviation becomes

$$\begin{aligned}\ddot{\mathbf{n}}^{i} &= -R_{\ j\ell m}^{i}u^{m}u^{j}n^{\ell} &(2)\\ &= -R_{\ t\ell t}^{i}n^{\ell} &(3)\end{aligned}$$

The relation between $\ddot{\mathbf{n}}^{i}$ and $\ddot{n}^{i}$ is given by

$$\ddot{\mathbf{n}}^{i}=\ddot{n}^{i}+\dot{n}^{k}u^{j}\Gamma_{\ kj}^{i}+n^{k}\dot{u}^{j}\Gamma_{\ kj}^{i}+n^{k}u^{j}\dot{\Gamma}_{\ kj}^{i}+\dot{n}^{k}u^{m}\Gamma_{\ km}^{i}+n^{\ell}u^{j}u^{m}\Gamma_{\ \ell j}^{k}\Gamma_{\ km}^{i} \tag{4}$$

As the Christoffel symbols are zero, the second, third, fifth and sixth terms are all zero. Also, as we're staying in a LIF (rather than just being in a LIF at one instant of time), the Christoffel symbols don't change with time, so $\dot{\Gamma}_{\ kj}^{i}=0$ and the fourth term vanishes as well. We're left with

$$\ddot{\mathbf{n}}^{i}=\ddot{n}^{i} \tag{5}$$

To an object at rest in the LIF, its coordinate time t is the same as its proper time τ so we can write this as

$$\frac{d^{2}n^{i}}{dt^{2}}=-R_{\ t\ell t}^{i}n^{\ell} \tag{6}$$

Note that although the Riemann tensor is defined in terms of the Christoffel symbols, the fact that all these symbols are zero in a LIF doesn't necessarily mean the Riemann tensor is also zero. The Riemann tensor is

$$R^i_{\ j\ell m} \equiv \partial_\ell \Gamma^i_{\ mj} - \partial_m \Gamma^i_{\ \ell j} + \Gamma^k_{\ mj} \Gamma^i_{\ \ell k} - \Gamma^k_{\ \ell j} \Gamma^i_{\ km} \tag{7}$$

Since it involves the derivatives of the Christoffel symbols, this in turn means we must have the second derivatives of g_{ij} and we've seen that these are not zero for curved spacetime.

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