

COVARIANT DERIVATIVE: COMMUTATIVITY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.8.

The second absolute gradient (or covariant derivative) of a four-vector is not commutative, as we can show by a direct derivation. Starting with the formula for the absolute gradient of a four-vector:

$$(1) \quad \nabla_j A^k \equiv \frac{\partial A^k}{\partial x^j} + A^i \Gamma_{ij}^k$$

and the formula for the absolute gradient of a mixed tensor:

$$(2) \quad \nabla_l C_j^i = \partial_l C_j^i + \Gamma_{lm}^i C_j^m - \Gamma_{lj}^m C_m^i$$

we can write out the second absolute gradient of a four-vector:

$$(3) \quad \nabla_i (\nabla_j A^k) = \partial_i \partial_j A^k + \Gamma_{jl}^k \partial_i A^l + A^\ell \partial_i \Gamma_{j\ell}^k - \Gamma_{ji}^m (\partial_m A^k + A^\ell \Gamma_{m\ell}^k) + \Gamma_{im}^k (\partial_j A^m + A^\ell \Gamma_{j\ell}^m)$$

If we now swap i and j , we get, using the commutativity of ordinary derivatives and the symmetry of Γ_{ji}^m :

$$(4) \quad \nabla_j (\nabla_i A^k) = \partial_j \partial_i A^k + \Gamma_{il}^k \partial_j A^l + A^\ell \partial_j \Gamma_{i\ell}^k - \Gamma_{ji}^m (\partial_m A^k + A^\ell \Gamma_{m\ell}^k) + \Gamma_{jm}^k (\partial_i A^m + A^\ell \Gamma_{i\ell}^m)$$

Subtracting these two equations gives

$$(5) \quad (\nabla_i \nabla_j - \nabla_j \nabla_i) A^k = \left(\partial_i \Gamma_{j\ell}^k - \partial_j \Gamma_{i\ell}^k + \Gamma_{im}^k \Gamma_{j\ell}^m - \Gamma_{jm}^k \Gamma_{i\ell}^m \right) A^\ell$$

Using the definition of the Riemann tensor:

$$(6) \quad R_{j\ell m}^i \equiv \partial_\ell \Gamma_{mj}^i - \partial_m \Gamma_{\ell j}^i + \Gamma_{mj}^k \Gamma_{\ell k}^i - \Gamma_{\ell j}^k \Gamma_{km}^i$$

we have

$$(7) \quad (\nabla_i \nabla_j - \nabla_j \nabla_i) A^k = R^k{}_{\ell ij} A^\ell$$

Thus the covariant derivative commutes only if the Riemann tensor is zero, which occurs only in flat spacetime.