COVARIANT DERIVATIVE: COMMUTATIVITY

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 18; Problem P18.8.

The second absolute gradient (or covariant derivative) of a four-vector is not commutative, as we can show by a direct derivation. Starting with the formula for the absolute gradient of a four-vector:

$$\nabla_j A^k \equiv \frac{\partial A^k}{\partial x^j} + A^i \Gamma^k_{\ ij} \tag{0.1}$$

and the formula for the absolute gradient of a mixed tensor:

$$\nabla_l C^i_j = \partial_l C^i_j + \Gamma^i_{lm} C^m_j - \Gamma^m_{lj} C^i_m \tag{0.2}$$

we can write out the second absolute gradient of a four-vector:

(0.3)
$$\nabla_i \left(\nabla_j A^k\right) = \partial_i \partial_j A^k + \Gamma^k_{j\ell} \partial_i A^\ell + A^\ell \partial_i \Gamma^k_{j\ell} - \Gamma^m_{ji} \left(\partial_m A^k + A^\ell \Gamma^k_{m\ell}\right) + \Gamma^k_{im} \left(\partial_j A^m + A^\ell \Gamma^m_{j\ell}\right)$$

If we now swap i and j, we get, using the commutativity of ordinary derivatives and the symmetry of Γ^m_{ji}:

(0.4)
$$\nabla_j \left(\nabla_i A^k\right) = \partial_i \partial_j A^k + \Gamma^k_{i\ell} \partial_j A^\ell + A^\ell \partial_j \Gamma^k_{i\ell} - \Gamma^m_{ji} \left(\partial_m A^k + A^\ell \Gamma^k_{m\ell}\right) + \Gamma^k_{jm} \left(\partial_i A^m + A^\ell \Gamma^m_{i\ell}\right)$$

Subtracting these two equations gives

$$\left(\nabla_i \nabla_j - \nabla_j \nabla_i\right) A^k = \left(\partial_i \Gamma^k_{j\ell} - \partial_j \Gamma^k_{i\ell} + \Gamma^k_{im} \Gamma^m_{j\ell} - \Gamma^k_{jm} \Gamma^m_{i\ell}\right) A^\ell \tag{0.5}$$

Using the definition of the Riemann tensor:

$$R^i_{\ j\ell m} \equiv \partial_\ell \Gamma^i_{\ mj} - \partial_m \Gamma^i_{\ \ell j} + \Gamma^k_{\ mj} \Gamma^i_{\ \ell k} - \Gamma^k_{\ \ell j} \Gamma^i_{\ km} \tag{0.6}$$

we have

$$\left(\nabla_i \nabla_j - \nabla_j \nabla_i\right) A^k = R^k_{\ \ell i j} A^\ell \tag{0.7}$$

Thus the covariant derivative commutes only if the Riemann tensor is zero, which occurs only in flat spacetime.