

## RIEMANN TENSOR: SYMMETRIES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Boxes 19.1, 19.2.

We can derive a few useful symmetries of the Riemann tensor by looking at its form in a locally inertial frame (LIF). At the origin of such a frame, all first derivatives of  $g_{ij}$  are zero, which means the Christoffel symbols are all zero there. However, the second derivatives of  $g_{ij}$  are not, in general, zero, so the derivatives of the Christoffel symbols will not, in general, be zero either.

Using the definition of the Riemann tensor:

$$R^i{}_{j\ell m} \equiv \partial_\ell \Gamma^i{}_{mj} - \partial_m \Gamma^i{}_{\ell j} + \Gamma^k{}_{mj} \Gamma^i{}_{\ell k} - \Gamma^k{}_{\ell j} \Gamma^i{}_{km} \quad (1)$$

we can write it at the origin of a LIF:

$$R^i{}_{j\ell m} \equiv \partial_\ell \Gamma^i{}_{mj} - \partial_m \Gamma^i{}_{\ell j} \quad (2)$$

The Christoffel symbols are

$$\Gamma^m{}_{ij} = \frac{1}{2} g^{ml} (\partial_j g_{il} + \partial_i g_{lj} - \partial_l g_{ji}) \quad (3)$$

The symmetries of the Riemann tensor are easiest to write if we look at its form with all indices lowered, that is:

$$R_{nj\ell m} = g_{nk} R^k{}_{j\ell m} \quad (4)$$

$$= g_{nk} (\partial_\ell \Gamma^k{}_{mj} - \partial_m \Gamma^k{}_{\ell j}) \quad (5)$$

First, we calculate the derivative:

$$\partial_\ell \Gamma^k{}_{mj} = \frac{1}{2} \partial_\ell g^{ki} (\partial_j g_{mi} + \partial_m g_{ij} - \partial_i g_{jm}) + \frac{1}{2} g^{ki} (\partial_\ell \partial_j g_{mi} + \partial_\ell \partial_m g_{ij} - \partial_\ell \partial_i g_{jm}) \quad (6)$$

At the origin of a LIF, the first term is zero since all first derivatives of  $g_{ij}$  are zero, so we're left with

$$\partial_\ell \Gamma^k{}_{mj} = \frac{1}{2} g^{ki} (\partial_\ell \partial_j g_{mi} + \partial_\ell \partial_m g_{ij} - \partial_\ell \partial_i g_{jm}) \quad (7)$$

Multiplying this by  $g_{kn}$  and using  $g_{kn}g^{ik} = \delta_n^i$ , we have

$$g_{kn}\partial_\ell\Gamma_{mj}^k = \frac{1}{2}\delta_n^i(\partial_\ell\partial_jg_{mi} + \partial_\ell\partial_mg_{ij} - \partial_\ell\partial_ig_{jm}) \quad (8)$$

$$= \frac{1}{2}(\partial_\ell\partial_jg_{mn} + \partial_\ell\partial_mg_{nj} - \partial_\ell\partial_n g_{jm}) \quad (9)$$

By substituting indices, we can get the second term in 5:

$$g_{nk}\partial_m\Gamma_{lj}^k = \frac{1}{2}(\partial_m\partial_jg_{ln} + \partial_m\partial_\ell g_{nj} - \partial_m\partial_n g_{jl}) \quad (10)$$

Subtracting 10 from 9 we see that the middle terms cancel, so we're left with

$$\boxed{R_{nj\ell m} = \frac{1}{2}(\partial_\ell\partial_jg_{mn} + \partial_m\partial_n g_{j\ell} - \partial_\ell\partial_n g_{jm} - \partial_m\partial_j g_{\ell n})} \quad (11)$$

This equation is valid only at the origin on a LIF.

From this we can get some symmetry properties. First, if we interchange the first two indices  $n$  and  $j$  in the tensor we see that the first and third terms on the RHS in 11 swap, as do the second and fourth, so we end up with the negative of what we started with. That is

$$\boxed{R_{jn\ell m} = -R_{nj\ell m}} \quad (12)$$

If we interchange the last two indices  $\ell$  and  $m$ , again the first term swaps with the fourth, and the second with the third, so we get the same result:

$$\boxed{R_{njm\ell} = -R_{nj\ell m}} \quad (13)$$

If we swap the first and third indices, and also the second and fourth, we get

$$R_{\ell mnj} = \frac{1}{2}(\partial_n\partial_m g_{j\ell} + \partial_j\partial_\ell g_{mn} - \partial_n\partial_\ell g_{mj} - \partial_j\partial_m g_{n\ell}) \quad (14)$$

$$= R_{nj\ell m} \quad (15)$$

Thus the Riemann tensor is symmetric under interchange of its first two indices with its last two:

$$\boxed{R_{\ell mnj} = R_{nj\ell m}} \quad (16)$$

A final symmetry property is a bit more subtle. If we cyclically permute the last 3 indices  $j$ ,  $\ell$  and  $m$  and add up the 3 terms, we get

$$\begin{aligned}
R_{nj\ell m} + R_{n\ell m j} + R_{nmj\ell} &= \frac{1}{2} (\partial_\ell \partial_j g_{mn} + \partial_m \partial_n g_{j\ell} - \partial_\ell \partial_n g_{jm} - \partial_m \partial_j g_{\ell n}) + \\
&\frac{1}{2} (\partial_m \partial_\ell g_{jn} + \partial_j \partial_n g_{\ell m} - \partial_m \partial_n g_{\ell j} - \partial_j \partial_\ell g_{mn}) + \\
&\frac{1}{2} (\partial_j \partial_m g_{\ell n} + \partial_\ell \partial_n g_{mj} - \partial_j \partial_n g_{m\ell} - \partial_\ell \partial_m g_{jn})
\end{aligned} \tag{17}$$

Using the symmetry of  $g_{ij} = g_{ji}$  and the fact that partial derivatives commute, we find that the first two terms in the first line cancel with the last two terms in the second line, the first two in the second line cancel with the last two in the third line, and the first two in the third line cancel with the last two in the first line, giving the result:

$$\boxed{R_{nj\ell m} + R_{n\ell m j} + R_{nmj\ell} = 0} \tag{18}$$

We've derived these results for the special case at the origin of a LIF. However, the origin of a LIF defines one particular event in spacetime and since all these symmetries are tensor equations, they must be true for that particular event, regardless of which coordinate system we're using. Further, in our discussion of LIFs, we showed that we could define a LIF with its origin at *any* point in spacetime, provided that point is locally flat (that is, that there is no singularity at that point). So the argument shows that these symmetries are true for all non-singular points in spacetime.

Incidentally, it might be confusing that we can say that these symmetries are universally valid at all points in all coordinate systems just because they are tensor equations, while we say that 11 is valid only at the origin of a LIF. The difference is that 11 is written explicitly in terms of a particular metric  $g_{ij}$  and that metric is defined precisely so that all its first derivatives are zero at the origin of the LIF. If we wanted an equation for  $R_{nj\ell m}$  at some *other* point in spacetime, we could write it in the same form, but we'd need to find a *different* metric  $g_{ij}$  whose first derivatives are zero at this other point. If we wanted to use the original metric, then since this other point is not at the origin of the original LIF, the  $\Gamma_{k\ell}^j$  would not be zero at this point since the derivatives of  $g_{ij}$  wouldn't be zero there, and the expression for  $R_{nj\ell m}$  would be more complicated in terms of the original metric.

#### PINGBACKS

Pingback: Riemann tensor in 2-d polar coordinates

Pingback: Ricci tensor and curvature scalar

Pingback: Riemann tensor: counting independent components

Pingback: The Bianchi identity for the Riemann tensor

- Pingback: Riemann tensor for surface of a sphere
- Pingback: Ricci tensor and curvature scalar for a sphere
- Pingback: Riemann tensor in the Schwarzschild metric: observer's view
- Pingback: Falling into a black hole: tidal forces
- Pingback: Einstein equation: trying the Ricci tensor as a solution
- Pingback: Einstein equation in the Newtonian limit
- Pingback: Einstein tensor and Einstein equation
- Pingback: Gravity can't exist in 2 spacetime dimensions
- Pingback: Gravity can't exist in 3 spacetime dimensions either
- Pingback: Einstein equation for an exponential metric
- Pingback: Plane symmetric spacetime