RIEMANN TENSOR: SYMMETRIES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Boxes 19.1, 19.2.

We can derive a few useful symmetries of the Riemann tensor by looking at its form in a locally inertial frame (LIF). At the origin of such a frame, all first derivatives of g_{ij} are zero, which means the Christoffel symbols are all zero there. However, the second derivatives of g_{ij} are not, in general, zero, so the derivatives of the Christoffel symbols will not, in general, be zero either.

Using the definition of the Riemann tensor:

$$R^{i}{}_{j\ell m} \equiv \partial_{\ell}\Gamma^{i}{}_{mj} - \partial_{m}\Gamma^{i}{}_{\ell j} + \Gamma^{k}{}_{mj}\Gamma^{i}{}_{\ell k} - \Gamma^{k}{}_{\ell j}\Gamma^{i}{}_{km} \tag{1}$$

we can write it at the origin of a LIF:

$$R^{i}{}_{j\ell m} \equiv \partial_{\ell}\Gamma^{i}{}_{mj} - \partial_{m}\Gamma^{i}{}_{\ell j} \tag{2}$$

The Christoffel symbols are

$$\Gamma^{m}_{ij} = \frac{1}{2}g^{ml}\left(\partial_{j}g_{il} + \partial_{i}g_{lj} - \partial_{l}g_{ji}\right) \tag{3}$$

The symmetries of the Riemann tensor are easiest to write if we look at its form with all indices lowered, that is:

$$\begin{aligned} R_{nj\ell m} &= g_{nk}R^{k}{}_{j\ell m} && (4)\\ &= g_{nk}\left(\partial_{\ell}\Gamma^{k}{}_{mj} - \partial_{m}\Gamma^{k}{}_{\ell j}\right) && (5) \end{aligned}$$

First, we calculate the derivative:

$$\partial_{\ell}\Gamma^{k}{}_{mj} = \frac{1}{2}\partial_{\ell}g^{ki}\left(\partial_{j}g_{mi} + \partial_{m}g_{ij} - \partial_{i}g_{jm}\right) + \frac{1}{2}g^{ki}\left(\partial_{\ell}\partial_{j}g_{mi} + \partial_{\ell}\partial_{m}g_{ij} - \partial_{\ell}\partial_{i}g_{jm}\right) \tag{6}$$

At the origin of a LIF, the first term is zero since all first derivatives of g_{ij} are zero, so we're left with

$$\partial_\ell \Gamma^k_{\ mj} = \frac{1}{2} g^{ki} \left(\partial_\ell \partial_j g_{mi} + \partial_\ell \partial_m g_{ij} - \partial_\ell \partial_i g_{jm} \right) \tag{7}$$

Multiplying this by g_{kn} and using $g_{kn} g^{ik} = \delta^i_n$, we have

$$g_{kn} \partial_\ell \Gamma^k_{\ mj} = \frac{1}{2} \delta^i_n \left(\partial_\ell \partial_j g_{mi} + \partial_\ell \partial_m g_{ij} - \partial_\ell \partial_i g_{jm} \right) \tag{8}$$

$$= \frac{1}{2} \left(\partial_\ell \partial_j g_{mn} + \partial_\ell \partial_m g_{nj} - \partial_\ell \partial_n g_{jm} \right) \tag{9}$$

By substituting indices, we can get the second term in 5:

$$g_{nk} \partial_m \Gamma^k_{\ \ell j} = \frac{1}{2} \left(\partial_m \partial_j g_{\ell n} + \partial_m \partial_\ell g_{nj} - \partial_m \partial_n g_{j\ell} \right) \tag{10}$$

Subtracting 10 from 9 we see that the middle terms cancel, so we're left with

$$\boxed{R_{nj\ell m} = \frac{1}{2} \left(\partial_\ell \partial_j g_{mn} + \partial_m \partial_n g_{j\ell} - \partial_\ell \partial_n g_{jm} - \partial_m \partial_j g_{\ell n} \right)} \tag{11}$$

This equation is valid only at the origin on a LIF.

From this we can get some symmetry properties. First, if we interchange the first two indices n and j in the tensor we see that the first and third terms on the RHS in 11 swap, as do the second and fourth, so we end up with the negative of what we started with. That is

$$\boxed{R_{jn\ell m} = -R_{nj\ell m}} \tag{12}$$

If we interchange the last two indices ℓ and m, again the first term swaps with the fourth, and the second with the third, so we get the same result:

$$\boxed{R_{njm\ell} = -R_{nj\ell m}} \tag{13}$$

If we swap the first and third indices, and also the second and fourth, we get

$$R_{\ell mnj} = \frac{1}{2} \left(\partial_n \partial_m g_{j\ell} + \partial_j \partial_\ell g_{mn} - \partial_n \partial_\ell g_{mj} - \partial_j \partial_m g_{n\ell} \right) \tag{14}$$

$$= R_{nj\ell m} \tag{15}$$

Thus the Riemann tensor is symmetric under interchange of its first two indices with its last two:

$$\boxed{R_{\ell mnj} = R_{nj\ell m}} \tag{16}$$

A final symmetry property is a bit more subtle. If we cyclically permute the last 3 indices j, ℓ and m and add up the 3 terms, we get

$$\begin{aligned} R_{nj\ell m} + R_{n\ell mj} + R_{nmj\ell} = & \frac{1}{2}\left(\partial_\ell \partial_j g_{mn} + \partial_m \partial_n g_{j\ell} - \partial_\ell \partial_n g_{jm} - \partial_m \partial_j g_{\ell n}\right) + \\ & \frac{1}{2}\left(\partial_m \partial_\ell g_{jn} + \partial_j \partial_n g_{\ell m} - \partial_m \partial_n g_{\ell j} - \partial_j \partial_\ell g_{mn}\right) + \\ & \frac{1}{2}\left(\partial_j \partial_m g_{\ell n} + \partial_\ell \partial_n g_{mj} - \partial_j \partial_n g_{m\ell} - \partial_\ell \partial_m g_{jn}\right) \end{aligned} \tag{17}$$

Using the symmetry of $g_{ij} = g_{ji}$ and the fact that partial derivatives commute, we find that the first two terms in the first line cancel with the last two terms in the second line, the first two in the second line cancel with the last two in the third line, and the first two in the third line cancel with the last two in the first line, giving the result:

$$\boxed{R_{nj\ell m} + R_{n\ell mj} + R_{nmj\ell} = 0} \tag{18}$$

We've derived these results for the special case at the origin of a LIF. However, the origin of a LIF defines one particular event in spacetime and since all these symmetries are tensor equations, they must be true for that particular event, regardless of which coordinate system we're using. Further, in our discussion of LIFs, we showed that we could define a LIF with its origin at *any* point in spacetime, provided that point is locally flat (that is, that there is no singularity at that point). So the argument shows that these symmetries are true for all non-singular points in spacetime.

Incidentally, it might be confusing that we can say that these symmetries are universally valid at all points in all coordinate systems just because they are tensor equations, while we say that 11 is valid only at the origin of a LIF. The difference is that 11 is written explicitly in terms of a particular metric g_{ij} and that metric is defined precisely so that all its first derivatives are zero at the origin of the LIF. If we wanted an equation for $R_{nj\ell m}$ at some *other* point in spacetime, we could write it in the same form, but we'd need to find a *different* metric g_{ij} whose first derivatives are zero at this other point. If we wanted to use the original metric, then since this other point is not at the origin of the original LIF, the $\Gamma^j_{k\ell}$ would not be zero at this point since the derivatives of g_{ij} wouldn't be zero there, and the expression for $R_{nj\ell m}$ would be more complicated in terms of the original metric.

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