RIEMANN TENSOR: COUNTING COMPONENTS IN GENERAL

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Problems 19.2, 19.3.

We can generalize the method for counting the number of independent components in the Riemann tensor to n-dimensional spacetime. As before, we know that the first pair and last pair of indices must both consist of different values in order for the component to be (possibly) non-zero. With n components to choose from, this gives us $\binom{n}{2}^2 = \left[\frac{1}{2}n(n-1)\right]^2$ components. If we arrange these components in a $\frac{1}{2}n(n-1)\times\frac{1}{2}n(n-1)$ matrix with the rows and columns labelled by the first and second pairs of indices, respectively, then due to the condition

$$R_{nj\ell m} = R_{\ell mnj} \tag{1}$$

the lower triangle of this matrix is a mirror of the upper triangle, so the possible number of independent components is reduced to at most $\sum_{i=1}^{n(n-1)/2} = \frac{1}{2}\left(\frac{1}{2}n(n-1)\right)\left(\frac{1}{2}n(n-1)+1\right)$. This gives

$$\frac{1}{2}\left(\frac{1}{2}n(n-1)\right)\left(\frac{1}{2}n(n-1)+1\right) = \frac{1}{8}n(n-1)\left[n(n-1)+2\right] \tag{2}$$

We now need to apply the final symmetry condition, which is

$$R_{nj\ell m} + R_{n\ell mj} + R_{nmj\ell} = 0 \tag{3}$$

As we saw in the last post, this equation gives new constraints only if all four indices are different, and the order in which these indices appear in the first term doesn't matter. Therefore, this equation provides a total of $\binom{n}{4} = \frac{1}{24}n(n-1)(n-2)(n-3)$ constraints, so the total number of independent components is

$$
\begin{aligned}
N(n) &= \frac{1}{8}n(n-1)[n(n-1)+2]-\frac{1}{24}n(n-1)(n-2)(n-3) && (4)\\
&= \frac{3}{24}\left(n^2-n\right)\left(n^2-n+2\right)-\frac{1}{24}\left(n^2-n\right)\left(n^2-5n+6\right) && (5)\\
&= \frac{1}{24}\left(n^2-n\right)\left(2n^2+2n\right) && (6)\\
&= \frac{1}{12}\left(n^2-n\right)\left(n^2+n\right) && (7)\\
&= \frac{1}{12}\left(n^4-n^2\right) && (8)\\
&= \frac{1}{12}n^2\left(n^2-1\right) && (9)
\end{aligned}
$$

This formula works even if $n<4$, since the second term in 4 is zero in this case.

The numbers of independent components for the first few dimensions are

n	$N(n)$
2	1
3	6
4	20
5	50
6	105