RIEMANN TENSOR: COUNTING INDEPENDENT COMPONENTS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Box 19.3.

The symmetries of the Riemann tensor mean that only some of its components are independent. The two conditions

$$R_{jn\ell m} = -R_{nj\ell m} \tag{1}$$

$$R_{njm\ell} = -R_{nj\ell m} \tag{2}$$

show that all components where either the first and second indices, or the third and fourth indices are equal must be zero. In four dimensional spacetime, this means that at most $\binom{4}{2}^2 = 36$ components can be non-zero, since we can choose two distinct values for both the first and last pairs of indices.

The condition

$$R_{nj\ell m} = R_{\ell mnj} \tag{3}$$

means that $R_{nl\ell m}$ is symmetric with respect to its two pairs of indices. If we arrange the 36 non-zero components in a 6×6 matrix where the rows and columns are labelled by the distinct pairs of values 01, 02, 03, 12, 13, 23, then the lower triangle of this matrix is the mirror image of the upper triangle, meaning we can eliminate $\sum_{i=1}^{5} i = 15$ more components, leaving $36-15=21$ possibly independent components.

There is one final symmetry condition for the Riemann tensor, and it is the trickiest to handle.

$$R_{nj\ell m} + R_{n\ell mj} + R_{nmj\ell} = 0 \tag{4}$$

The first thing we need to show about this condition is that if any two indices are equal, then 4 follows from the other three conditions and tells us nothing new. To see this, consider the various ways in which two indices can be equal.

First, suppose $n=j$. Then $R_{nn\ell m}=0$ from 1, so we're left with

$$R_{n\ell mn}+R_{nmn\ell} = R_{n\ell mn}+R_{n\ell nm} \tag{5}$$
$$= R_{n\ell mn}-R_{n\ell mn} \tag{6}$$
$$= 0 \tag{7}$$

The first line uses 3 and the second line uses 2. Because 4 cyclically permutes the last three indices, the same argument also applies to the cases $n=\ell$ and $n=m$.

Now suppose $j=\ell$. Then the third term in 4 is $R_{nmjj}=0$ using 2, so we're left with

$$R_{njjm}+R_{njmj}=R_{njjm}-R_{njjm}=0 \tag{8}$$

using 2.

For $j=m$ the second term in 4 is $R_{n\ell jj}=0$ so

$$R_{nj\ell j}+R_{njj\ell}=R_{nj\ell j}-R_{nj\ell j}=0 \tag{9}$$

Finally, if $\ell=m$, the first term in 4 is $R_{njmm}=0$ and

$$R_{nmmj}+R_{nmjm}=R_{nmmj}-R_{nmmj}=0 \tag{10}$$

Now that we know that 4 gives us new information only if all the indices are different, how many ways can we choose these indices? Given that we have four indices, we might think that there are $4!=24$ possibilities, depending on the ordering of the indices. In fact, for any set of four indices, the condition gives us only one independent constraint, as the four different indices can be placed in any order in the first term. Starting with 4 with all indices different, suppose we put j first instead of n. We can do this by swapping n and j to get

$$R_{jn\ell m}+R_{j\ell mn}+R_{jmn\ell} = -R_{nj\ell m}+R_{mnj\ell}+R_{n\ell jm} \tag{11}$$
$$= -R_{nj\ell m}-R_{nmj\ell}-R_{n\ell mj} \tag{12}$$
$$= 0 \tag{13}$$

We used the first three symmetries in the first and second lines and then 4 to get the zero in the third line.

If we swap n with ℓ we get

$$R_{\ell jnm}+R_{\ell nmj}+R_{\ell mjn} = R_{nm\ell j}-R_{n\ell mj}+R_{jn\ell m} \tag{14}$$
$$= -R_{nmj\ell}-R_{n\ell mj}-R_{nj\ell m} \tag{15}$$
$$= 0 \tag{16}$$

If we swap n and m we get

$$R_{mj\ell n}+R_{m\ell nj}+R_{mnj\ell} = R_{\ell nmj}+R_{njm\ell}-R_{nmj\ell} \tag{17}$$
$$= -R_{n\ell mj}-R_{nj\ell m}-R_{nmj\ell} \tag{18}$$
$$= 0 \tag{19}$$

Swapping any two of the last three indices gives the same result, since these three indices are present in a cyclic permutation, so we need to consider only one such case, say swapping j with m:

$$R_{nm\ell j}+R_{n\ell jm}+R_{njm\ell} = -R_{nmj\ell}-R_{n\ell mj}-R_{nj\ell m} \tag{20}$$
$$= 0 \tag{21}$$

Therefore, the condition 4 can give us only $\binom{4}{4}=1$ extra constraint. The total number of independent components in four-dimensional spacetime is therefore $21-1=20$.

Example. As another example, we can apply this reasoning to find the number of independent components in two dimensions. First, the number of possible pairs with distinct values is $\binom{2}{2}=1$, so the matrix referred to above is only 1×1. To verify that the condition 4 doesn't reduce the number of independent components any further, note that with only 2 possible values for indices, we must repeat at least one of them when choosing the indices in $R_{nj\ell m}$, so this condition doesn't in fact tell us anything new. Thus there is only one independent component in two dimensions.

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