THE BIANCHI IDENTITY FOR THE RIEMANN TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Box 19.4.

Another relation of the Riemann tensor involves the covariant derivative of the tensor, and is known as the Bianchi identity (actually the second Bianchi identity; the first identity is the symmetry relation $R_{nj\ell m}+R_{n\ell mj}+R_{nmj\ell}=0$ that we saw earlier). The identity is easiest to derive at the origin of a locally inertial frame (LIF), where the first derivatives of the metric tensor, and thus the Christoffel symbols, are all zero. At this point, we have

$$R_{nj\ell m}=\frac{1}{2}\left(\partial_\ell\partial_j g_{mn}+\partial_m\partial_n g_{j\ell}-\partial_\ell\partial_n g_{jm}-\partial_m\partial_j g_{\ell n}\right) \tag{1}$$

If the Christoffel symbols are all zero, then the covariant derivative becomes the ordinary derivative

$$\nabla_j A^k \equiv \partial_j A^k + A^i\Gamma^k_{ij} = \partial_j A^k \tag{2}$$

Therefore, we get, at the origin of a LIF:

$$\begin{aligned}
\nabla_k R_{nj\ell m} &= \partial_k R_{nj\ell m} &(3)\\
&= \frac{1}{2}\left(\partial_k\partial_\ell\partial_j g_{mn}+\partial_k\partial_m\partial_n g_{j\ell}-\partial_k\partial_\ell\partial_n g_{jm}-\partial_k\partial_m\partial_j g_{\ell n}\right) &(4)
\end{aligned}$$

By cyclically permuting the index of the derivative with the last two indices of the tensor, we get

$$\begin{aligned}
\nabla_\ell R_{njmk} &= \partial_\ell R_{njmk} &(5)\\
&= \frac{1}{2}\left(\partial_\ell\partial_m\partial_j g_{kn}+\partial_\ell\partial_k\partial_n g_{jm}-\partial_\ell\partial_m\partial_n g_{jk}-\partial_\ell\partial_k\partial_j g_{mn}\right) &(6)\\
\nabla_m R_{njk\ell} &= \partial_m R_{njk\ell} &(7)\\
&= \frac{1}{2}\left(\partial_m\partial_k\partial_j g_{\ell n}+\partial_m\partial_\ell\partial_n g_{jk}-\partial_m\partial_k\partial_n g_{j\ell}-\partial_m\partial_\ell\partial_j g_{kn}\right) &(8)
\end{aligned}$$

By adding up 4, 6 and 8 and using the commutativity of partial derivatives, we see that the terms cancel in pairs, so we get

$$\boxed{\nabla_k R_{nj\ell m}+\nabla_\ell R_{njmk}+\nabla_m R_{njk\ell}=0} \tag{9}$$

As usual we can use the argument that since we can set up a LIF with its origin at any non-singular point in spacetime, this equation is true everywhere and since the covariant derivative is a tensor, this is a tensor equation and is thus valid in all coordinate systems. This is the Bianchi identity.

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