

THE BIANCHI IDENTITY FOR THE RIEMANN TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Box 19.4.

Another relation of the Riemann tensor involves the covariant derivative of the tensor, and is known as the Bianchi identity (actually the second Bianchi identity; the first identity is the symmetry relation $R_{njlm} + R_{nlmj} + R_{nmjl} = 0$ that we saw earlier). The identity is easiest to derive at the origin of a locally inertial frame (LIF), where the first derivatives of the metric tensor, and thus the Christoffel symbols, are all zero. At this point, we have

$$(0.1) \quad R_{njlm} = \frac{1}{2} (\partial_\ell \partial_j g_{mn} + \partial_m \partial_n g_{j\ell} - \partial_\ell \partial_n g_{jm} - \partial_m \partial_j g_{\ell n})$$

If the Christoffel symbols are all zero, then the covariant derivative becomes the ordinary derivative

$$(0.2) \quad \nabla_j A^k \equiv \partial_j A^k + A^i \Gamma_{ij}^k = \partial_j A^k$$

Therefore, we get, at the origin of a LIF:

$$(0.3) \quad \nabla_k R_{njlm} = \partial_k R_{njlm}$$

$$(0.4) \quad = \frac{1}{2} (\partial_k \partial_\ell \partial_j g_{mn} + \partial_k \partial_m \partial_n g_{j\ell} - \partial_k \partial_\ell \partial_n g_{jm} - \partial_k \partial_m \partial_j g_{\ell n})$$

By cyclically permuting the index of the derivative with the last two indices of the tensor, we get

$$(0.5) \quad \nabla_\ell R_{njmk} = \partial_\ell R_{njmk}$$

$$(0.6) \quad = \frac{1}{2} (\partial_\ell \partial_m \partial_j g_{kn} + \partial_\ell \partial_k \partial_n g_{jm} - \partial_\ell \partial_m \partial_n g_{jk} - \partial_\ell \partial_k \partial_j g_{mn})$$

$$(0.7) \quad \nabla_m R_{njkl} = \partial_m R_{njkl}$$

$$(0.8) \quad = \frac{1}{2} (\partial_m \partial_k \partial_j g_{\ell n} + \partial_m \partial_\ell \partial_n g_{jk} - \partial_m \partial_k \partial_n g_{j\ell} - \partial_m \partial_\ell \partial_j g_{kn})$$

By adding up 0.4, 0.6 and 0.8 and using the commutativity of partial derivatives, we see that the terms cancel in pairs, so we get

$$(0.9) \quad \boxed{\nabla_k R_{nj\ell m} + \nabla_\ell R_{njmk} + \nabla_m R_{njkl} = 0}$$

As usual we can use the argument that since we can set up a LIF with its origin at any non-singular point in spacetime, this equation is true everywhere and since the covariant derivative is a tensor, this is a tensor equation and is thus valid in all coordinate systems. This is the Bianchi identity.

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