

RIEMANN TENSOR FOR SURFACE OF A SPHERE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Problem 19.4.

As an example of the Riemann tensor in 2-d curved space we can use our old standby of the surface of a sphere. As usual, we need the Christoffel symbols and we get them by comparing the two forms of the geodesic equation.

$$(0.1) \quad \frac{d}{d\tau} (g_{aj}\dot{x}^j) - \frac{1}{2}\partial_a g_{ij}\dot{x}^i\dot{x}^j = 0$$

$$(0.2) \quad \ddot{x}^m + \Gamma_{ij}^m\dot{x}^j\dot{x}^i = 0$$

where as usual a dot denotes a derivative with respect to proper time τ .

For a sphere, the interval is

$$(0.3) \quad ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

Note that r (the radius of the sphere) is a constant here.

From 0.1 we get, with $a = \theta$:

$$(0.4) \quad r^2\ddot{x}^j - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0$$

Dividing through by r^2 and comparing with 0.2 we get

$$(0.5) \quad \Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta$$

$$(0.6) \quad \Gamma_{\theta\phi}^{\theta} = \Gamma_{\phi\theta}^{\theta} = \Gamma_{\theta\theta}^{\theta} = 0$$

With $a = \phi$ we have

$$(0.7) \quad 2r^2 \sin \theta \cos \theta \dot{\theta}\dot{\phi} + r^2 \sin^2 \theta \ddot{\phi} = 0$$

$$(0.8) \quad 2 \cot \theta \dot{\theta}\dot{\phi} + \ddot{\phi} = 0$$

$$(0.9) \quad \Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta$$

$$(0.10) \quad \Gamma_{\theta\theta}^{\phi} = \Gamma_{\phi\phi}^{\phi} = 0$$

We can use these results to get the Riemann tensor. Unfortunately, in the form R^a_{bcd} , the Riemann tensor doesn't have all the symmetries of the form R_{abcd} , so if we want the latter form, we need to work out the former form first and then use

$$(0.11) \quad R_{abcd} = g_{af} R^f_{bcd}$$

$$(0.12) \quad = g_{af} \left(\partial_c \Gamma^f_{db} - \partial_d \Gamma^f_{cb} + \Gamma^k_{db} \Gamma^f_{ck} - \Gamma^k_{cb} \Gamma^f_{kd} \right)$$

Although we know there is only one independent component in 2-d, we can work out all four non-zero components to see how the calculations go.

$$(0.13) \quad R_{\theta\phi\theta\phi} = g_{\theta f} R^f_{\phi\theta\phi}$$

$$(0.14) \quad = g_{\theta\theta} R^\theta_{\phi\theta\phi}$$

$$(0.15) \quad = r^2 \left(\partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\theta\phi} + \Gamma^k_{\phi\phi} \Gamma^\theta_{\theta k} - \Gamma^k_{\theta\phi} \Gamma^\theta_{k\phi} \right)$$

$$(0.16) \quad = r^2 \left(\sin^2 \theta - \cos^2 \theta - 0 + 0 + \cos^2 \theta \right)$$

$$(0.17) \quad = r^2 \sin^2 \theta$$

$$(0.18) \quad R_{\theta\phi\phi\theta} = g_{\theta\theta} R^\theta_{\phi\phi\theta}$$

$$(0.19) \quad = r^2 \left(\partial_\phi \Gamma^\theta_{\theta\phi} - \partial_\theta \Gamma^\theta_{\phi\phi} + \Gamma^k_{\theta\phi} \Gamma^\theta_{\phi k} - \Gamma^k_{\phi\phi} \Gamma^\theta_{k\theta} \right)$$

$$(0.20) \quad = -R_{\theta\phi\theta\phi}$$

$$(0.21) \quad = -r^2 \sin^2 \theta$$

$$(0.22) \quad R_{\phi\theta\theta\phi} = g_{\phi\phi} R^\phi_{\theta\theta\phi}$$

$$(0.23) \quad = r^2 \sin^2 \theta \left(\partial_\theta \Gamma^\phi_{\phi\theta} - \partial_\phi \Gamma^\phi_{\theta\theta} + \Gamma^k_{\phi\theta} \Gamma^\phi_{\theta k} - \Gamma^k_{\theta\theta} \Gamma^\phi_{k\phi} \right)$$

$$(0.24) \quad = r^2 \sin^2 \theta \left(-\frac{1}{\sin^2 \theta} - 0 + 0 + \frac{\cos^2 \theta}{\sin^2 \theta} \right)$$

$$(0.25) \quad = -r^2 \sin^2 \theta$$

$$(0.26) \quad R_{\phi\theta\phi\theta} = g_{\phi\phi} R^\phi_{\theta\phi\theta}$$

$$(0.27) \quad = r^2 \sin^2 \theta \left(\partial_\phi \Gamma^\phi_{\theta\theta} - \partial_\theta \Gamma^\phi_{\phi\theta} + \Gamma^k_{\theta\theta} \Gamma^\phi_{k\phi} - \Gamma^k_{\phi\theta} \Gamma^\phi_{\theta k} \right)$$

$$(0.28) \quad = -R_{\phi\theta\theta\phi}$$

$$(0.29) \quad = r^2 \sin^2 \theta$$

Finally, we can calculate one of the other components to verify that it's zero.

$$(0.30) \quad R_{\theta\theta\theta\theta} = g_{\theta\theta} R^{\theta}_{\theta\theta\theta}$$

$$(0.31) \quad = r^2 \left(\partial_{\theta} \Gamma^{\theta}_{\theta\theta} - \partial_{\theta} \Gamma^{\theta}_{\theta\theta} + \Gamma^k_{\theta\theta} \Gamma^{\theta}_{k\theta} - \Gamma^k_{\theta\theta} \Gamma^{\theta}_{\theta k} \right)$$

$$(0.32) \quad = 0$$

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