RIEMANN TENSOR FOR SURFACE OF A SPHERE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Problem 19.4.

As an example of the Riemann tensor in 2-d curved space we can use our old standby of the surface of a sphere. As usual, we need the Christoffel symbols and we get them by comparing the two forms of the geodesic equation.

$$\frac{d}{d\tau}\left(g_{aj}\dot{x}^j\right)-\frac{1}{2}\partial_a g_{ij}\dot{x}^i\dot{x}^j = 0 \tag{1}$$

$$\ddot{x}^m+\Gamma_{ij}^m\dot{x}^j\dot{x}^i = 0 \tag{2}$$

where as usual a dot denotes a derivative with respect to proper time τ.

For a sphere, the interval is

$$ds^2=r^2d\theta^2+r^2\sin^2\theta d\phi^2 \tag{3}$$

Note that r (the radius of the sphere) is a constant here.

From 1 we get, with $a=\theta$:

$$r^2\ddot{x}^j-r^2\sin\theta\cos\theta\dot{\phi}^2=0 \tag{4}$$

Dividing through by r^2 and comparing with 2 we get

$$\Gamma_{\phi\phi}^\theta = -\sin\theta\cos\theta \tag{5}$$

$$\Gamma_{\theta\phi}^\theta = \Gamma_{\phi\theta}^\theta=\Gamma_{\theta\theta}^\theta=0 \tag{6}$$

With $a=\phi$ we have

$$2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi}+r^2\sin^2\theta\ddot{\phi} = 0 \tag{7}$$

$$2\cot\theta\dot{\theta}\dot{\phi}+\ddot{\phi} = 0 \tag{8}$$

$$\Gamma_{\theta\phi}^\phi=\Gamma_{\phi\theta}^\phi = \cot\theta \tag{9}$$

$$\Gamma_{\theta\theta}^\phi=\Gamma_{\phi\phi}^\phi = 0 \tag{10}$$

We can use these results to get the Riemann tensor. Unfortunately, in the form $R^a_{\ bcd}$, the Riemann tensor doesn't have all the symmetries of the form R_{abcd}, so if we want the latter form, we need to work out the former form first and then use

$$R_{abcd} = g_{af} R^f_{\ bcd} \tag{11}$$
$$= g_{af}\left(\partial_c \Gamma^f_{\ db} - \partial_d \Gamma^f_{\ cb} + \Gamma^k_{\ db}\Gamma^f_{\ ck} - \Gamma^k_{\ cb}\Gamma^f_{\ kd}\right) \tag{12}$$

Although we know there is only one independent component in 2-d, we can work out all four non-zero components to see how the calculations go.

$$R_{\theta\phi\theta\phi} = g_{\theta f} R^f_{\ \phi\theta\phi} \tag{13}$$
$$= g_{\theta\theta} R^\theta_{\ \phi\theta\phi} \tag{14}$$
$$= r^2\left(\partial_\theta \Gamma^\theta_{\ \phi\phi} - \partial_\phi \Gamma^\theta_{\ \theta\phi} + \Gamma^k_{\ \phi\phi}\Gamma^\theta_{\ \theta k} - \Gamma^k_{\ \theta\phi}\Gamma^\theta_{\ k\phi}\right) \tag{15}$$
$$= r^2\left(\sin^2\theta - \cos^2\theta - 0 + 0 + \cos^2\theta\right) \tag{16}$$
$$= r^2\sin^2\theta \tag{17}$$
$$R_{\theta\phi\phi\theta} = g_{\theta\theta} R^\theta_{\ \phi\phi\theta} \tag{18}$$
$$= r^2\left(\partial_\phi \Gamma^\theta_{\ \theta\phi} - \partial_\theta \Gamma^\theta_{\ \phi\phi} + \Gamma^k_{\ \theta\phi}\Gamma^\theta_{\ \phi k} - \Gamma^k_{\ \phi\phi}\Gamma^\theta_{\ k\theta}\right) \tag{19}$$
$$= -R_{\theta\phi\theta\phi} \tag{20}$$
$$= -r^2\sin^2\theta \tag{21}$$
$$R_{\phi\theta\theta\phi} = g_{\phi\phi} R^\phi_{\ \theta\theta\phi} \tag{22}$$
$$= r^2\sin^2\theta\left(\partial_\theta \Gamma^\phi_{\ \phi\theta} - \partial_\phi \Gamma^\phi_{\ \theta\theta} + \Gamma^k_{\ \phi\theta}\Gamma^\phi_{\ \theta k} - \Gamma^k_{\ \theta\theta}\Gamma^\phi_{\ k\phi}\right) \tag{23}$$
$$= r^2\sin^2\theta\left(-\frac{1}{\sin^2\theta} - 0 + 0 + \frac{\cos^2\theta}{\sin^2\theta}\right) \tag{24}$$
$$= -r^2\sin^2\theta \tag{25}$$
$$R_{\phi\theta\phi\theta} = g_{\phi\phi} R^\phi_{\ \theta\phi\theta} \tag{26}$$
$$= r^2\sin^2\theta\left(\partial_\phi \Gamma^\phi_{\ \theta\theta} - \partial_\theta \Gamma^\phi_{\ \phi\theta} + \Gamma^k_{\ \theta\theta}\Gamma^\phi_{\ k\phi} - \Gamma^k_{\ \phi\theta}\Gamma^\phi_{\ \theta k}\right) \tag{27}$$
$$= -R_{\phi\theta\theta\phi} \tag{28}$$
$$= r^2\sin^2\theta \tag{29}$$

Finally, we can calculate one of the other components to verify that it's zero.

$$R_{\theta\theta\theta\theta} = g_{\theta\theta} R^{\theta}_{\ \theta\theta\theta} \tag{30}$$

$$= r^2 \left(\partial_\theta \Gamma^{\theta}_{\ \theta\theta} - \partial_\theta \Gamma^{\theta}_{\ \theta\theta} + \Gamma^{k}_{\ \theta\theta} \Gamma^{\theta}_{\ k\theta} - \Gamma^{k}_{\ \theta\theta} \Gamma^{\theta}_{\ \theta k} \right) \tag{31}$$

$$= 0 \tag{32}$$

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