

## RIEMANN TENSOR FOR SURFACE OF A SPHERE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Problem 19.4.

As an example of the Riemann tensor in 2-d curved space we can use our old standby of the surface of a sphere. As usual, we need the Christoffel symbols and we get them by comparing the two forms of the geodesic equation.

$$\frac{d}{d\tau} (g_{aj}\dot{x}^j) - \frac{1}{2}\partial_a g_{ij}\dot{x}^i\dot{x}^j = 0 \quad (1)$$

$$\ddot{x}^m + \Gamma_{ij}^m\dot{x}^j\dot{x}^i = 0 \quad (2)$$

where as usual a dot denotes a derivative with respect to proper time  $\tau$ .

For a sphere, the interval is

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (3)$$

Note that  $r$  (the radius of the sphere) is a constant here.

From 1 we get, with  $a = \theta$ :

$$r^2\ddot{x}^j - r^2 \sin \theta \cos \theta \dot{\phi}^2 = 0 \quad (4)$$

Dividing through by  $r^2$  and comparing with 2 we get

$$\Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta \quad (5)$$

$$\Gamma_{\theta\phi}^{\theta} = \Gamma_{\phi\theta}^{\theta} = \Gamma_{\theta\theta}^{\theta} = 0 \quad (6)$$

With  $a = \phi$  we have

$$2r^2 \sin \theta \cos \theta \dot{\theta}\dot{\phi} + r^2 \sin^2 \theta \ddot{\phi} = 0 \quad (7)$$

$$2 \cot \theta \dot{\theta}\dot{\phi} + \ddot{\phi} = 0 \quad (8)$$

$$\Gamma_{\theta\phi}^{\phi} = \Gamma_{\phi\theta}^{\phi} = \cot \theta \quad (9)$$

$$\Gamma_{\theta\theta}^{\phi} = \Gamma_{\phi\phi}^{\phi} = 0 \quad (10)$$

We can use these results to get the Riemann tensor. Unfortunately, in the form  $R^a_{bcd}$ , the Riemann tensor doesn't have all the symmetries of the form  $R_{abcd}$ , so if we want the latter form, we need to work out the former form first and then use

$$R_{abcd} = g_{af} R^f_{bcd} \quad (11)$$

$$= g_{af} \left( \partial_c \Gamma^f_{db} - \partial_d \Gamma^f_{cb} + \Gamma^k_{db} \Gamma^f_{ck} - \Gamma^k_{cb} \Gamma^f_{kd} \right) \quad (12)$$

Although we know there is only one independent component in 2-d, we can work out all four non-zero components to see how the calculations go.

$$R_{\theta\phi\theta\phi} = g_{\theta f} R^f_{\phi\theta\phi} \quad (13)$$

$$= g_{\theta\theta} R^\theta_{\phi\theta\phi} \quad (14)$$

$$= r^2 \left( \partial_\theta \Gamma^\theta_{\phi\phi} - \partial_\phi \Gamma^\theta_{\theta\phi} + \Gamma^k_{\phi\phi} \Gamma^\theta_{\theta k} - \Gamma^k_{\theta\phi} \Gamma^\theta_{k\phi} \right) \quad (15)$$

$$= r^2 \left( \sin^2 \theta - \cos^2 \theta - 0 + 0 + \cos^2 \theta \right) \quad (16)$$

$$= r^2 \sin^2 \theta \quad (17)$$

$$R_{\theta\phi\phi\theta} = g_{\theta\theta} R^\theta_{\phi\phi\theta} \quad (18)$$

$$= r^2 \left( \partial_\phi \Gamma^\theta_{\theta\phi} - \partial_\theta \Gamma^\theta_{\phi\phi} + \Gamma^k_{\theta\phi} \Gamma^\theta_{\phi k} - \Gamma^k_{\phi\phi} \Gamma^\theta_{k\theta} \right) \quad (19)$$

$$= -R_{\theta\phi\theta\phi} \quad (20)$$

$$= -r^2 \sin^2 \theta \quad (21)$$

$$R_{\phi\theta\theta\phi} = g_{\phi\phi} R^\phi_{\theta\theta\phi} \quad (22)$$

$$= r^2 \sin^2 \theta \left( \partial_\theta \Gamma^\phi_{\phi\theta} - \partial_\phi \Gamma^\phi_{\theta\theta} + \Gamma^k_{\phi\theta} \Gamma^\phi_{\theta k} - \Gamma^k_{\theta\theta} \Gamma^\phi_{k\phi} \right) \quad (23)$$

$$= r^2 \sin^2 \theta \left( -\frac{1}{\sin^2 \theta} - 0 + 0 + \frac{\cos^2 \theta}{\sin^2 \theta} \right) \quad (24)$$

$$= -r^2 \sin^2 \theta \quad (25)$$

$$R_{\phi\theta\phi\theta} = g_{\phi\phi} R^\phi_{\theta\phi\theta} \quad (26)$$

$$= r^2 \sin^2 \theta \left( \partial_\phi \Gamma^\phi_{\theta\theta} - \partial_\theta \Gamma^\phi_{\phi\theta} + \Gamma^k_{\theta\theta} \Gamma^\phi_{k\phi} - \Gamma^k_{\phi\theta} \Gamma^\phi_{\theta k} \right) \quad (27)$$

$$= -R_{\phi\theta\theta\phi} \quad (28)$$

$$= r^2 \sin^2 \theta \quad (29)$$

Finally, we can calculate one of the other components to verify that it's zero.

$$R_{\theta\theta\theta\theta} = g_{\theta\theta}R^{\theta}_{\theta\theta\theta} \quad (30)$$

$$= r^2 \left( \partial_{\theta}\Gamma^{\theta}_{\theta\theta} - \partial_{\theta}\Gamma^{\theta}_{\theta\theta} + \Gamma^k_{\theta\theta}\Gamma^{\theta}_{k\theta} - \Gamma^k_{\theta\theta}\Gamma^{\theta}_{\theta k} \right) \quad (31)$$

$$= 0 \quad (32)$$

## PINGBACKS

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