RIEMANN TENSOR IN 2-D FLAT SPACE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Problem P19.5.

Another example of the Riemann tensor in a 2-d space. The metric is

$$ds^2 = dp^2 + \frac{dq^2}{b^2q^2} \tag{1}$$

where b is a constant. The metric tensor is therefore $g_{pp} = 1$, $g_{qq} = 1/b^2q^2$. By comparing the two forms of the geodesic equation, we can calculate the Christoffel symbols.

$$\begin{aligned} g_{aj}\ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j\dot{x}^i &= 0 \qquad (2)\\ \ddot{x}^a + \Gamma^a_{\ ij}\dot{x}^j\dot{x}^i &= 0 \qquad (3)\end{aligned}$$

With $a = p$, we get from 2

$$\ddot{p} = 0 \tag{4}$$

From 3, we see that $\Gamma^p_{\ ij} = 0$ for all i and j.

With $a = q$, we have

$$\begin{aligned} \frac{1}{b^2q^2}\ddot{q} - \frac{2}{b^2q^3}\dot{q}^2 + \frac{1}{b^2q^3}\dot{q}^2 &= 0 \qquad (5)\\ \ddot{q} - \frac{1}{q}\dot{q}^2 &= 0 \qquad (6)\end{aligned}$$

Comparing with 3 we find

$$\begin{aligned} \Gamma^q_{\ qq} &= -\frac{1}{q} \qquad (7)\\ \Gamma^q_{\ pq} &= \Gamma^q_{\ qp} = \Gamma^q_{\ pp} = 0 \qquad (8)\end{aligned}$$

The only independent component of the Riemann tensor in 2-d is $R^p_{\ qpq}$:

$$\begin{aligned} R^{p}{}_{qpq} &= \partial_p \Gamma^{p}{}_{qq} - \partial_q \Gamma^{p}{}_{pq} + \Gamma^{p}{}_{kp} \Gamma^{k}{}_{qq} - \Gamma^{p}{}_{qk} \Gamma^{k}{}_{pq} \qquad (9) \\ &= 0 \qquad (10) \end{aligned}$$

since all terms involve components of the form Γ^{p}_{ij}. Therefore, the Ricci tensor is also zero: $R_{ij} = 0$ as is the curvature scalar $R = 0$. The space is flat.