RICCI TENSOR AND CURVATURE SCALAR FOR A SPHERE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Box 19.6.

As an example of calculating the Ricci tensor and curvature scalar we'll find them for the 2-d surface of a sphere. The Ricci tensor is calculated from the Riemann tensor, and that in turn depends on the Christoffel symbols, so we'll need them first. It's easiest to find them from the geodesic equation

$$g_{aj}\ddot{x}^j+\left(\partial_i g_{aj}-\frac{1}{2}\partial_a g_{ij}\right)\dot{x}^j\dot{x}^i=0 \tag{1}$$

which is formally equivalent to

$$\ddot{x}^m+\Gamma_{ij}^m\dot{x}^j\dot{x}^i=0 \tag{2}$$

The metric for a sphere is

$$ds^2=r^2d\theta^2+r^2\sin^2\theta d\phi^2 \tag{3}$$

where r is the constant radius of the sphere, so

$$g_{\theta\theta} = r^2 \tag{4}$$

$$g_{\phi\phi} = r^2\sin^2\theta \tag{5}$$

We have two equations arising from 1. For $a=\theta$

$$r^2\ddot{\theta}-r^2\sin\theta\cos\theta\dot{\phi}^2=0 \tag{6}$$

Comparing with 2 we get, after dividing out the r^2:

$$\Gamma_{\phi\phi}^{\theta}=-\sin\theta\cos\theta=-\frac{1}{2}\sin 2\theta \tag{7}$$

For $a=\phi$:

$$r^2\sin^2\theta\ddot{\phi}^2+2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi}=0 \tag{8}$$

Dividing through by $r^2\sin^2\theta$ and comparing with 2 we get (remember that the second term is $\left(\Gamma^{\phi}_{\ \theta\phi}+\Gamma^{\phi}_{\ \phi\theta}\right)\dot{\theta}\dot{\phi}$ and that $\Gamma^{\phi}_{\ \theta\phi}=\Gamma^{\phi}_{\ \phi\theta}$):

$$\Gamma^{\phi}_{\ \theta\phi}=\Gamma^{\phi}_{\ \phi\theta}=\cot\theta \tag{9}$$

All other Christoffel symbols are zero.

In 2D, the Riemann tensor has only one independent component, which we can take to be $R_{\theta\phi\theta\phi}$, which can be calculated from

$$R^{i}_{\ j\ell m}\equiv\partial_\ell\Gamma^{i}_{\ mj}-\partial_m\Gamma^{i}_{\ \ell j}+\Gamma^{k}_{\ mj}\Gamma^{i}_{\ \ell k}-\Gamma^{k}_{\ \ell j}\Gamma^{i}_{\ km} \tag{10}$$

Lowering the first index, we have

$$
\begin{aligned}
R_{\theta\phi\theta\phi} &= g_{\theta i}R^{i}_{\ \phi\theta\phi} && (11)\\
&= g_{\theta\theta}R^{\theta}_{\ \phi\theta\phi} && (12)\\
&= r^2\left(\partial_\theta\Gamma^{\theta}_{\ \phi\phi}-\partial_\phi\Gamma^{\theta}_{\ \theta\phi}+\Gamma^{k}_{\ \phi\phi}\Gamma^{\theta}_{\ \theta k}-\Gamma^{k}_{\ \theta\phi}\Gamma^{\theta}_{\ k\phi}\right) && (13)\\
&= r^2\left(-\cos 2\theta-0+0-\Gamma^{\phi}_{\ \theta\phi}\Gamma^{\theta}_{\ \phi\phi}\right) && (14)\\
&= r^2\left(\sin^2\theta-\cos^2\theta+\cos^2\theta\right) && (15)\\
&= r^2\sin^2\theta && (16)
\end{aligned}
$$

We can now find the Ricci tensor.

$$R_{ij}=g^{ab}R_{aibj} \tag{17}$$

Since the metric is diagonal and g^{ab} is the inverse of g_{ab}, we have

$$
\begin{aligned}
g^{\theta\theta} &= \frac{1}{g^{\theta\theta}}=\frac{1}{r^2} && (18)\\
g^{\phi\phi} &= \frac{1}{g^{\phi\phi}}=\frac{1}{r^2\sin^2\theta} && (19)
\end{aligned}
$$

so, using the symmetries of the Riemann tensor,

$$R_{\theta\theta} = g^{ab}R_{a\theta b\theta} \tag{20}$$
$$= g^{\phi\phi}R_{\phi\theta\phi\theta} \tag{21}$$
$$= \frac{1}{r^2\sin^2\theta}r^2\sin^2\theta \tag{22}$$
$$= 1 \tag{23}$$
$$R_{\phi\phi} = g^{ab}R_{a\phi b\phi} \tag{24}$$
$$= g^{\theta\theta}R_{\theta\phi\theta\phi} \tag{25}$$
$$= \sin^2\theta \tag{26}$$
$$R_{\theta\phi} = R_{\phi\theta} = 0 \tag{27}$$

We can get the upstairs version of the Ricci tensor as well:

$$R^{\theta\theta} = g^{\theta i}g^{\theta j}R_{ij} \tag{28}$$
$$= g^{\theta\theta}g^{\theta\theta}R_{\theta\theta} \tag{29}$$
$$= \frac{1}{r^4} \tag{30}$$
$$R^{\phi\phi} = g^{\phi\phi}g^{\phi\phi}R_{\phi\phi} \tag{31}$$
$$= \frac{1}{r^4\sin^2\theta} \tag{32}$$

The curvature scalar is

$$R = g_{ij}R^{ij} \tag{33}$$
$$= r^2\frac{1}{r^4} + r^2\sin^2\theta\frac{1}{r^4\sin^2\theta} \tag{34}$$
$$= \frac{2}{r^2} \tag{35}$$

As we would expect, the curvature of a sphere decreases as its radius gets larger.

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