

RIEMANN TENSOR IN A 2-D CURVED SPACE

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Problem P19.6.

Here's another example of the Riemann tensor in a 2-d coordinate system. The tensor is

$$(1) \quad R^i{}_{j\ell m} \equiv \partial_\ell \Gamma^i{}_{mj} - \partial_m \Gamma^i{}_{\ell j} + \Gamma^k{}_{mj} \Gamma^i{}_{\ell k} - \Gamma^k{}_{\ell j} \Gamma^i{}_{km}$$

As usual, we need the Christoffel symbols, which we can get by comparing the two forms of the geodesic equation. These equations are

$$(2) \quad g_{aj} \ddot{x}^j + \left(\partial_i g_{aj} - \frac{1}{2} \partial_a g_{ij} \right) \dot{x}^j \dot{x}^i = 0$$

$$(3) \quad \ddot{x}^a + \Gamma^a{}_{ij} \dot{x}^j \dot{x}^i = 0$$

The metric is

$$(4) \quad ds^2 = dp^2 + e^{2p/p_0} dq^2$$

so $g_{pp} = 1$ and $g_{qq} = e^{2p/p_0}$. For the two coordinates, 2 gives us

$$(5) \quad \ddot{p} - \frac{1}{p_0} e^{2p/p_0} \dot{q}^2 = 0$$

$$(6) \quad e^{2p/p_0} \ddot{q} + \frac{2}{p_0} e^{2p/p_0} \dot{p} \dot{q} = 0$$

Dividing through by the coefficient of the second derivative in the second equation case gives:

$$(7) \quad \ddot{p} - \frac{1}{p_0} e^{2p/p_0} \dot{q}^2 = 0$$

$$(8) \quad \ddot{q} + \frac{2}{p_0} \dot{p} \dot{q} = 0$$

Comparing with 3 we get

$$(9) \quad \Gamma_{qq}^p = -\frac{1}{p_0} e^{2p/p_0}$$

$$(10) \quad \Gamma_{pq}^q = \Gamma_{qp}^q = \frac{1}{p_0}$$

with all other Christoffel symbols equal to zero.

The only independent Riemann tensor component in 2-d is R_{ppq}^p :

$$(11) \quad R_{ppq}^p = \partial_p \Gamma_{qq}^p - \partial_q \Gamma_{pq}^p + \Gamma_{kp}^p \Gamma_{qq}^k - \Gamma_{qk}^p \Gamma_{pq}^k$$

$$(12) \quad = \partial_p \Gamma_{qq}^p - 0 + 0 - \Gamma_{qq}^p \Gamma_{pq}^q$$

$$(13) \quad = -\frac{2}{p_0^2} e^{2p/p_0} + \frac{1}{p_0^2} e^{2p/p_0}$$

$$(14) \quad = -\frac{1}{p_0^2} e^{2p/p_0}$$

Any non-zero component indicates that the space is curved, so this metric represents a curved space.