RIEMANN TENSOR IN A 2-D CURVED SPACE

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Problem P19.6.

Here's another example of the Riemann tensor in a 2-d coordinate system. The tensor is

$$R^{i}{}_{j\ell m} \equiv \partial_{\ell}\Gamma^{i}{}_{mj} - \partial_{m}\Gamma^{i}{}_{\ell j} + \Gamma^{k}{}_{mj}\Gamma^{i}{}_{\ell k} - \Gamma^{k}{}_{\ell j}\Gamma^{i}{}_{km} \tag{0.1}$$

As usual, we need the Christoffel symbols, which we can get by comparing the two forms of the geodesic equation. These equations are

$$g_{aj}\ddot{x}^{j} + \left(\partial_{i}g_{aj} - \frac{1}{2}\partial_{a}g_{ij}\right)\dot{x}^{j}\dot{x}^{i} = 0 \tag{0.2}$$

$$\ddot{x}^{a} + \Gamma^{a}_{ij}\dot{x}^{j}\dot{x}^{i} = 0 \tag{0.3}$$

The metric is

$$ds^{2} = dp^{2} + e^{2p/p_{0}}dq^{2} \tag{0.4}$$

so $g_{pp} = 1$ and $g_{qq} = e^{2p/p_0}$. For the two coordinates, 0.2 gives us

$$\ddot{p} - \frac{1}{p_{0}}e^{2p/p_{0}}\dot{q}^{2} = 0 \tag{0.5}$$

$$e^{2p/p_{0}}\ddot{q} + \frac{2}{p_{0}}e^{2p/p_{0}}\dot{p}\dot{q} = 0 \tag{0.6}$$

Dividing through by the coefficient of the second derivative in the second equation case gives:

$$\ddot{p} - \frac{1}{p_{0}}e^{2p/p_{0}}\dot{q}^{2} = 0 \tag{0.7}$$

$$\ddot{q} + \frac{2}{p_{0}}\dot{p}\dot{q} = 0 \tag{0.8}$$

Comparing with 0.3 we get

$$\Gamma^{p}_{qq} = -\frac{1}{p_0}e^{2p/p_0} \tag{0.9}$$

$$\Gamma^{q}_{pq} = \Gamma^{q}_{qp} = \frac{1}{p_0} \tag{0.10}$$

with all other Christoffel symbols equal to zero.

The only independent Riemann tensor component in 2-d is R^{p}_{qpq} :

$$R^{p}_{qpq} = \partial_p\Gamma^{p}_{qq} - \partial_q\Gamma^{p}_{pq} + \Gamma^{p}_{kp}\Gamma^{k}_{qq} - \Gamma^{p}_{qk}\Gamma^{k}_{pq} \tag{0.11}$$

$$= \partial_p\Gamma^{p}_{qq} - 0 + 0 - \Gamma^{p}_{qq}\Gamma^{q}_{pq} \tag{0.12}$$

$$= -\frac{2}{p_0^2}e^{2p/p_0} + \frac{1}{p_0^2}e^{2p/p_0} \tag{0.13}$$

$$= -\frac{1}{p_0^2}e^{2p/p_0} \tag{0.14}$$

Any non-zero component indicates that the space is curved, so this metric represents a curved space.