RIEMANN TENSOR FOR 3-D SPHERICAL COORDINATES

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Problem 19.7.

We saw that the Riemann tensor for the surface of a sphere had a non-zero component, indicating that this is a curved space. If we use spherical coordinates in 3-d space, however, the Riemann tensor should be zero, since this is a flat space.

As usual, we need the Christoffel symbols and we get them by comparing the two forms of the geodesic equation.

$$\begin{aligned} \frac{d}{d\tau}\left(g_{aj}\dot{x}^j\right)-\frac{1}{2}\partial_a g_{ij}\dot{x}^i\dot{x}^j &= 0 \quad (1)\\ \ddot{x}^m+\Gamma_{ij}^m\dot{x}^j\dot{x}^i &= 0 \quad (2)\end{aligned}$$

where as usual a dot denotes a derivative with respect to proper time τ.

For spherical coordinates, the interval is

$$ds^2=dr^2+r^2d\theta^2+r^2\sin^2\theta d\phi^2 \quad (3)$$

Note that r is now a variable, rather than the constant radius of the sphere in the 2-d system.

From 1 we get, with $a=\theta$:

$$2r\dot{r}\dot{\theta}+r^2\ddot{\theta}-r^2\sin\theta\cos\theta\dot{\phi}^2=0 \quad (4)$$

Dividing through by r^2 and comparing with 2 we get

$$\begin{aligned} \Gamma_{\phi\phi}^{\theta} &= -\sin\theta\cos\theta \quad (5)\\ \Gamma_{r\theta}^{\theta} &= \Gamma_{\theta r}^{\theta}=\frac{1}{r} \quad (6)\\ \Gamma_{\theta\phi}^{\theta} &= \Gamma_{\phi\theta}^{\theta}=\Gamma_{\theta\theta}^{\theta}=0 \quad (7)\end{aligned}$$

With $a=\phi$ we have

$$2r\sin^2\theta\dot{r}\dot{\phi}+2r^2\sin\theta\cos\theta\dot{\theta}\dot{\phi}+r^2\sin^2\theta\ddot{\phi}=0 \tag{8}$$
$$\frac{2}{r}\dot{r}\dot{\phi}+2\cot\theta\dot{\theta}\dot{\phi}+\ddot{\phi}=0 \tag{9}$$
$$\Gamma^{\phi}_{\theta\phi}=\Gamma^{\phi}_{\phi\theta}=\cot\theta \tag{10}$$
$$\Gamma^{\phi}_{r\phi}=\Gamma^{\phi}_{\phi r}=\frac{1}{r} \tag{11}$$
$$\Gamma^{\phi}_{\theta\theta}=\Gamma^{\phi}_{\phi\phi}=0 \tag{12}$$

For $a=r$, we have

$$\ddot{r}-\frac{1}{2}\left(2r\dot{\theta}^2+2r\sin^2\theta\dot{\phi}^2\right) = 0 \tag{13}$$
$$\ddot{r}-r\dot{\theta}^2-r\sin^2\theta\dot{\phi}^2 = 0 \tag{14}$$
$$\Gamma^{r}_{\theta\theta} = -r \tag{15}$$
$$\Gamma^{r}_{\phi\phi} = -r\sin^2\theta \tag{16}$$

We can use these results to get the Riemann tensor.

$$R_{abcd}=g_{af}R^{f}_{\ bcd} \tag{17}$$
$$=g_{af}\left(\partial_c\Gamma^{f}_{db}-\partial_d\Gamma^{f}_{cb}+\Gamma^{k}_{db}\Gamma^{f}_{ck}-\Gamma^{k}_{cb}\Gamma^{f}_{kd}\right) \tag{18}$$

Although we know there is only one independent component in 2-d, we can work out all four non-zero components to see how the calculations go.

$$R_{\theta\phi\theta\phi}=g_{\theta f}R^{f}_{\ \phi\theta\phi} \tag{19}$$
$$=g_{\theta\theta}R^{\theta}_{\ \phi\theta\phi} \tag{20}$$
$$=r^2\left(\partial_\theta\Gamma^{\theta}_{\phi\phi}-\partial_\phi\Gamma^{\theta}_{\theta\phi}+\Gamma^{k}_{\phi\phi}\Gamma^{\theta}_{\theta k}-\Gamma^{k}_{\theta\phi}\Gamma^{\theta}_{k\phi}\right) \tag{21}$$
$$=r^2\left(\sin^2\theta-\cos^2\theta-0-\frac{r\sin^2\theta}{r}+\cos^2\theta\right) \tag{22}$$
$$=0 \tag{23}$$

The other components of the Riemann tensor can be evaluated in the same way, and they all come out to zero, so spherical coordinates represent flat 3-d space.