RIEMANN TENSOR IN THE SCHWARZSCHILD METRIC: OBSERVER'S VIEW

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 19; Problem P19.8.

The component of a four-vector A along a basis vector $\mathbf{o}_i$ is given by the dot product of the four-vector with the basis vector. Since the dot product is a scalar, it can be computed in any reference frame. In particular, if we're dealing with a locally flat coordinate system with basis vectors $\mathbf{o}_i$ embedded in the Schwarzschild (S) metric, we can do the calculation in the global S metric since we know the components of $\mathbf{o}_i$ in the S frame. That is, the components of a four-vector A along each of the basis vectors is

$$A_{i,obs} = g_{jm}(\mathbf{o}_i)^j A^m \tag{1}$$

$$= (\mathbf{o}_i)^j A_j \tag{2}$$

The subscript *obs* indicates that the vector component is that seen by an observer in the locally flat frame with basis vectors $\mathbf{o}_i$.

We can extend this idea to find the components of any tensor in the locally flat frame, since we just apply the same procedure to each index of the tensor. For the Riemann tensor $R^i{}_{j\ell m}$ we would get

$$R_{ij\ell m,obs} = g_{ab}(\mathbf{o}_i)^a (\mathbf{o}_j)^c (\mathbf{o}_\ell)^d (\mathbf{o}_m)^e R^b{}_{cde} \tag{3}$$

To get the observer's tensor with the first index raised, we need to use the metric to raise the index. The correct metric to use is the metric of the locally flat frame, which is η^{ij}. Therefore we get

$$R^i{}_{j\ell m,obs} = \eta^{if} R_{fj\ell m,obs} \tag{4}$$

$$= \eta^{if} g_{ab}(\mathbf{o}_f)^a (\mathbf{o}_j)^c (\mathbf{o}_\ell)^d (\mathbf{o}_m)^e R^b{}_{cde} \tag{5}$$

For example, we can calculate the component $R^t{}_{ztz,obs}$ for a freely-falling observer by using the components for $\mathbf{o}_i$ that we worked out earlier.

$$R^t{}_{ztz,obs} = \eta^{tf} g_{ab}(\mathbf{o}_f)^a (\mathbf{o}_z)^c (\mathbf{o}_t)^d (\mathbf{o}_z)^e R^b{}_{cde} \tag{6}$$

To save typing, I'll write the unit vectors in normal type and without the parentheses, so this equation becomes

$$R^{t}{}_{ztz,obs}=\eta^{tf}g_{ab}o_f^a o_z^c o_t^d o_z^e R^{b}{}_{cde} \tag{7}$$

where sums are implied over all repeated indices except t and z.

Remember that a superscript on a basis vector refers to the component of that vector along the direction given by the superscript, and that this direction is one of those in the S metric. A subscript on a basis vector refers to the axis in the locally flat system along with that basis vector points. Thus o_z^r is the component along the r direction in the S metric of the vector $\mathbf{o}_z$ that points along the z direction in the locally flat system.

We'll reproduce here the basis vectors $\mathbf{o}_t$ and $\mathbf{o}_z$ for a freely falling observer (we won't need the other two vectors):

$$\mathbf{o}_t = \left[\left(1-\frac{2GM}{r}\right)^{-1},-\sqrt{\frac{2GM}{r}},0,0\right] \tag{8}$$

$$\mathbf{o}_z = \left[-\left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}},1,0,0\right] \tag{9}$$

Unfortunately, since both of these vectors have two non-zero components, the sums in 7 give us more than one term. In order to make use of the symmetries of the Riemann tensor, we'll rewrite 7 using $R_{acde}=g_{ab}R^{b}{}_{cde}$:

$$R^{t}{}_{ztz,obs}=\eta^{tf}o_f^a o_z^c o_t^d o_z^e R_{acde} \tag{10}$$

Since η^{ij} is diagonal, we must have $f=t$ and since $\eta^{tt}=-1$ we have

$$R^{t}{}_{ztz,obs}=-o_t^a o_z^c o_t^d o_z^e R_{acde} \tag{11}$$

Because of the symmetries, we must have $a\neq c$ and $d\neq e$, so there are four possibilities for these four indices:

$$[a,c,d,e]=[t,r,t,r],[r,t,r,t],[r,t,t,r],[t,r,r,t] \tag{12}$$

The sum in 11 thus expands to

$$R^{t}{}_{ztz,obs}=-R_{trtr}o_t^t o_z^r o_t^t o_z^r-R_{rtrt}o_t^r o_z^t o_t^r o_z^t-R_{rttr}o_t^r o_z^t o_t^t o_z^r-R_{trrt}o_t^t o_z^r o_t^r o_z^t \tag{13}$$

Using the symmetries when swapping the first two or the last two indices in the Riemann tensor in this lowered form, we can rewrite this as

$$R^{t}{}_{ztz,obs} = -R_{trtr}\left[o_t^t o_z^r o_t^t o_z^r + o_t^r o_z^t o_t^r o_z^t - o_t^r o_z^t o_t^t o_z^r - o_t^t o_z^r o_t^r o_z^t\right] \tag{14}$$

$$= -R_{trtr}\left[\left(o_t^t o_z^r\right)^2 + \left(o_t^r o_z^t\right)^2 - 2 o_t^r o_z^t o_t^t o_z^r\right] \tag{15}$$

$$= -R_{trtr}\left(o_t^t o_z^r - o_t^r o_z^t\right)^2 \tag{16}$$

$$= -R_{trtr}\left(\left(1-\frac{2GM}{r}\right)^{-1} - \left[-\left(1-\frac{2GM}{r}\right)^{-1}\sqrt{\frac{2GM}{r}}\right]\left[-\sqrt{\frac{2GM}{r}}\right]\right)^2 \tag{17}$$

$$= -R_{trtr}\left[\left(1-\frac{2GM}{r}\right)^{-1}\left(1-\frac{2GM}{r}\right)\right]^2 \tag{18}$$

$$= -R_{trtr} \tag{19}$$

We now need to find R_{trtr}, which is

$$R_{trtr} = g_{ta}R^{a}{}_{rtr} = g_{tt}R^{t}{}_{rtr} \tag{20}$$

We've worked out the last tensor component earlier, so plugging this in, we get

$$R^{t}{}_{ztz,obs} = -g_{tt}R^{t}{}_{rtr} \tag{21}$$

$$= \left(1-\frac{2GM}{r}\right)\left[\frac{2GM}{r^3}\left(1-\frac{2GM}{r}\right)^{-1}\right] \tag{22}$$

$$= \frac{2GM}{r^3} \tag{23}$$

Thus this component of the Riemann tensor has no singularity at $r = 2GM$ in the observer's local frame.

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