

## STRESS-ENERGY TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; Box 20.2.

We gave an argument that in general relativity, the source of gravity must be energy rather than rest mass. We'll refer to the energy density by the symbol  $\rho$ . We can determine the role of the energy density by analogy with our earlier argument about charge density.

To begin the analogy, we consider a collection of particles that are all at rest in some local inertial frame (LIF)  $S_0$ . These particles can be evenly distributed over some small volume, and the technical term in relativity for such a mass distribution is *dust*. In this frame, the energy density  $\rho_0$  is just equal to the mass density (since the particles aren't moving), so if each particle has rest mass  $m$ , we have

$$(1) \quad \rho_0 = n_0 m$$

where  $n_0$  is the number of particles per unit volume in the rest frame  $S_0$ .

If we now put ourselves in another LIF  $S$  that is moving at some uniform velocity  $\beta$  relative to the first frame then, due to Lorentz contraction, the volume containing the  $n_0$  particles is smaller by a factor of  $\gamma = 1/\sqrt{1-\beta^2}$ , so the numerical density of particles increases by  $\gamma$ :  $n = \gamma n_0$ . To find the total energy density measured by the observer in  $S$ , however, we need to take into account that the particles are also moving as seen in  $S$ , so they have kinetic energy as well as rest mass. The total energy of a particle of rest mass  $m$  moving at speed  $\beta$  is  $\gamma m$ , so the total energy density due to the combined effects of length contraction and kinetic energy is

$$(2) \quad \rho = \gamma n_0 \gamma m$$

We can write this in a more illuminating form if we recall the definition of four-velocity

$$(3) \quad u^i = \frac{dx^i}{d\tau}$$

and the relation between proper time and measured time in another LIF:

$$(4) \quad d\tau = \sqrt{1 - \beta^2} dt$$

In particular, the time component of the four-velocity is

$$(5) \quad u^t = \frac{dt}{d\tau} = \gamma$$

so 2 can be written as

$$(6) \quad \rho = n_0 m u^t u^t = \rho_0 u^t u^t$$

Now  $\rho_0$  is a relativistic scalar, since it is the energy density as measured in the dust's rest frame, so it is invariant. That is, every observer in every frame will agree that *if we measure the energy density of the dust in its rest frame*, it will have the value  $\rho_0$ . Therefore,  $\rho$  is the product of a scalar and two factors, each of which is a component of a four-vector. We can therefore propose that  $\rho$  is the  $tt$  component of a rank-2 tensor  $T^{ij}$  which we can define as

$$(7) \quad T^{ij} \equiv \rho_0 u^i u^j$$

From its definition, we see that  $T^{ij} = T^{ji}$  so the tensor is symmetric. This is called the *stress-energy tensor*, derived here for the special case of dust.

To see the physical meaning of each component of  $T^{ij}$ , we can start by noting that  $T^{tt}$  is the energy density as viewed by an observer in frame  $S$ . How about  $T^{tx}$ ? Using  $u^x = dx/d\tau = \gamma \frac{dx}{dt} = \gamma \beta_x$ , we have

$$(8) \quad T^{tx} = \rho_0 u^t u^x$$

$$(9) \quad = n_0 m u^t u^x$$

$$(10) \quad = \gamma n_0 m u^x$$

$$(11) \quad = n m u^x$$

$$(12) \quad = n \gamma m \beta_x$$

The term  $\gamma m \beta_x$  is the  $x$  component  $p^x$  of a single particle's momentum, so we can write this as

$$(13) \quad T^{tx} = n p^x$$

so that  $T^{tx}$  is the momentum density in the  $x$  direction.

Looked at another way, using  $p^t = \gamma m$  (the  $t$  component of the four-momentum is the particle's energy, so  $p^t$  is the energy of a single particle), we have

$$(14) \quad T^{tx} = np^t \beta_x$$

Now suppose we consider a small element of area  $A$  perpendicular to the  $x$  direction. In a small time  $dt$ , all particles within the volume  $A\beta_x dt$  will cross this area. Since the number density is  $n$ , the total number of particles that cross the area is  $nA\beta_x dt$  so we can write this as

$$(15) \quad T^{tx} = \frac{nA\beta_x dt}{Adt} p^t$$

That is,  $T^{tx}$  is the energy flux per unit area per unit time in the  $x$  direction. A similar argument applies to  $T^{ty}$  and  $T^{tz}$ .

For the other components, let's consider  $T^{kl}$  where  $k$  and  $l$  are spatial indices. In this case, we have

$$(16) \quad T^{kl} = \rho_0 u^k u^l$$

$$(17) \quad = n_0 m u^k u^l$$

$$(18) \quad = n_0 p^k \gamma \beta_l$$

$$(19) \quad = n \beta_l p^k$$

$$(20) \quad = \frac{nA\beta_l dt}{Adt} p^k$$

The first factor is the flux per unit area per unit time of particles in the  $l$  direction, so  $T^{kl}$  is the flux in the  $l$  direction of  $k$  momentum. For example,  $T^{xz}$  is the flux of  $z$  momentum in the  $x$  direction (or, since  $T$  is symmetric, the flux of  $x$  momentum in the  $z$  direction).

This might sound a bit confusing, but suppose we consider a particle with a velocity  $\beta$ , with components  $[\beta_x, \beta_y, \beta_z]$ . The component  $\beta_x$  (assuming it's positive, say) means the particle will cross an area element to its right after some time. However, because this particle also has momenta in the other two directions ( $y$  and  $z$ , assuming neither  $\beta_y$  nor  $\beta_z$  is zero), some  $y$  and  $z$  momenta are transported in the  $x$  direction.

The term *stress* in physics is distinct from ordinary *pressure*, since pressure refers just to a normal (perpendicular) force on a surface, while stress

allows parallel components of force as well. Thus if a particle strikes a surface at some angle other than normal, the force it exerts on the surface can have components both normal to and parallel to the surface.

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