STRESS-ENERGY TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; Box 20.2.

We gave an argument that in general relativity, the source of gravity must be energy rather than rest mass. We'll refer to the energy density by the symbol ρ. We can determine the role of the energy density by analogy with our earlier argument about charge density.

To begin the analogy, we consider a collection of particles that are all at rest in some local inertial frame (LIF) S_0. These particles can be evenly distributed over some small volume, and the technical term in relativity for such a mass distribution is *dust*. In this frame, the energy density ρ_0 is just equal to the mass density (since the particles aren't moving), so if each particle has rest mass m, we have

$$\rho_0 = n_0 m \tag{1}$$

where n_0 is the number of particles per unit volume in the rest frame S_0.

If we now put ourselves in another LIF S that is moving at some uniform velocity β relative to the first frame then, due to Lorentz contraction, the volume containing the n_0 particles is smaller by a factor of $\gamma = 1/\sqrt{1-\beta^2}$, so the numerical density of particles increases by γ: $n = \gamma n_0$. To find the total energy density measured by the observer in S, however, we need to take into account that the particles are also moving as seen in S, so they have kinetic energy as well as rest mass. The total energy of a particle of rest mass m moving at speed β is γm, so the total energy density due to the combined effects of length contraction and kinetic energy is

$$\rho = \gamma n_0 \gamma m \tag{2}$$

We can write this is a more illuminating form if we recall the definition of four-velocity

$$u^i = \frac{dx^i}{d\tau} \tag{3}$$

and the relation between proper time and measured time in another LIF:

$$d\tau = \sqrt{1-\beta^2}dt \tag{4}$$

In particular, the time component of the four-velocity is

$$u^t = \frac{dt}{d\tau} = \gamma \tag{5}$$

so 2 can be written as

$$\rho = n_0 m u^t u^t = \rho_0 u^t u^t \tag{6}$$

Now ρ_0 is a relativistic scalar, since it is the energy density as measured in the dust's rest frame, so it is invariant. That is, every observer in every frame will agree that *if we measure the energy density of the dust in its rest frame*, it will have the value ρ_0. Therefore, ρ is the product of a scalar and two factors, each of which is a component of a four-vector. We can therefore propose that ρ is the tt component of a rank-2 tensor T^{ij} which we can define as

$$T^{ij} \equiv \rho_0 u^i u^j \tag{7}$$

From its definition, we see that $T^{ij} = T^{ji}$ so the tensor is symmetric. This is called the *stress-energy tensor*, derived here for the special case of dust.

To see the physical meaning of each component of T^{ij}, we can start by noting that T^{tt} is the energy density as viewed by an observer is frame S. How about T^{tx}? Using $u^x = dx/d\tau = \gamma\frac{dx}{dt} = \gamma\beta_x$, we have

$$\begin{aligned} T^{tx} &= \rho_0 u^t u^x && (8)\\ &= n_0 m u^t u^x && (9)\\ &= \gamma n_0 m u^x && (10)\\ &= n m u^x && (11)\\ &= n\gamma m\beta_x && (12) \end{aligned}$$

The term $\gamma m\beta_x$ is the x component p^x of a single particle's momentum, so we can write this as

$$T^{tx} = np^x \tag{13}$$

so that T^{tx} is the momentum density in the x direction.

Looked at another way, using $p^t = \gamma m$ (the t component of the four-momentum is the particle's energy, so p^t is the energy of a single particle), we have

$$T^{tx}=np^{t}\beta_{x} \tag{14}$$

Now suppose we consider a small element of area A perpendicular to the x direction. In a small time dt, all particles within the volume $A\beta_{x}dt$ will cross this area. Since the number density is n, the total number of particles that cross the area is $nA\beta_{x}dt$ so we can write this as

$$T^{tx}=\frac{nA\beta_{x}dt}{Adt}p^{t} \tag{15}$$

That is, T^{tx} is the energy flux per unit area per unit time in the x direction. A similar argument applies to T^{ty} and T^{tz}.

For the other components, let's consider T^{kl} where k and l are spatial indices. In this case, we have

$$\begin{aligned}
T^{kl} &= \rho_{0}u^{k}u^{l} && (16)\\
&= n_{0}mu^{k}u^{l} && (17)\\
&= n_{0}p^{k}\gamma\beta_{l} && (18)\\
&= n\beta_{l}p^{k} && (19)\\
&= \frac{nA\beta_{l}dt}{Adt}p^{k} && (20)
\end{aligned}$$

The first factor is the flux per unit area per unit time of particles in the l direction, so T^{kl} is the flux in the l direction of k momentum. For example, T^{xz} is the flux of z momentum in the x direction (or, since T is symmetric, the flux of x momentum in the z direction).

This might sound a bit confusing, but suppose we consider a particle with a velocity β, with components $[\beta_{x},\beta_{y},\beta_{z}]$. The component β_{x} (assuming it's positive, say) means the particle will cross an area element to its right after some time. However, because this particle also has momenta in the other two directions (y and z, assuming neither β_{y} nor β_{z} is zero), some y and z momenta are transported in the x direction.

The term *stress* in physics is distinct from ordinary *pressure*, since pressure refers just to a normal (perpendicular) force on a surface, while stress allows parallel components of force as well. Thus if a particle strikes a surface at some angle other than normal, the force it exerts on the surface can have components both normal to and parallel to the surface.

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