STRESS-ENERGY TENSOR FOR A PERFECT FLUID AT REST

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; Box 20.3.

We've seen the stress-energy tensor T^{ij} for the special case of 'dust', that is a system in which we have a collection of particles that are all at rest relative to each other. In that special case,

$$T^{ij} \equiv \rho_0 u^i u^j \tag{1}$$

where ρ_0 is the energy density of the dust in its rest frame, and u^i is the four-velocity of a local inertial frame (LIF) moving relative to the dust.

Real fluids, of course, contain particles that are in some sort of random motion, even if the fluid as a whole is at rest. We can derive the stress-energy tensor for such fluids if we assume that they are *perfect* or *ideal* fluids, which means that the particles within the fluid don't interact with each other. This is still a simplification since in all real fluids, even gases, the particles do interact. In liquids, this interaction gives the fluids viscosity.

The idea is that we can build up a description of a perfect fluid by considering subsets of the particles where within each subset, the particles act like dust, that is, they all move at the same velocity. We can work out the contribution to T^{ij} from each of these subsets and then add them up to get the overall tensor.

To get started, consider the subset of particles that have four-velocity u^i as measured in the overall rest frame of the fluid (that is, the frame in which the average of the velocities of all the particles is zero). Then the contribution to T^{ij} from that subset is

$$\Delta T^{ij} = \rho_0 u^i u^j \tag{2}$$

where ρ_0 is now the density of particles within that subset only, and measured in that subset's rest frame. In terms of the number density n_0 (again mesured in the subset's frame) this is

$$\Delta T^{ij} = n_0 m u^i u^j \tag{3}$$

Due to length contraction, the number density as measured in the fluid's rest frame is

$$n=\frac{n_0}{\sqrt{1-\beta^2}}=\gamma n_0=u^t n_0 \tag{4}$$

where u^t is the time component of the four-velocity. Therefore we can write

$$\Delta T^{ij}=\frac{n}{u^t}mu^iu^j=\frac{nmu^imu^j}{mu^t}=n\frac{p^ip^j}{p^t} \tag{5}$$

where p^i is the four-momentum of a single particle as measured in the fluid's frame.

Now suppose we look at a cubic volume at rest in the fluid's frame with side length L (as measured in the fluid's frame; because of length contraction, the volume won't be a cube in any of the individual particle's frames). Then if N is the total number of particles in the subset, we have $n=N/L^3$ so

$$\Delta T^{ij}=\frac{N}{L^3}\frac{p^ip^j}{p^t} \tag{6}$$

The total stress-energy tensor is then the integral of this quantity over all of momentum space, since we need to add up contributions of all the subsets with each possible momentum. Since there will be different numbers of particles in each subset, the quantity N becomes a function of momentum: $N=N(\mathbf{p})$. If we assume that the motions of the particles are truly random, then it is equally likely that a particle with magnitude of momentum p is moving in any direction, so we'd expect $N(\mathbf{p})$ to be a function of the *magnitude* of $\mathbf{p}$ only, rather than on its direction as well. That is, we have

$$\begin{aligned} N &= N(p) && (7)\\ p &= \sqrt{(p^x)^2+(p^y)^2+(p^z)^2} && (8)\end{aligned}$$

and the total stress-energy tensor is

$$T^{ij}=\frac{1}{L^3}\int\int\int N(p)\frac{p^ip^j}{p^t}dp^xdp^ydp^z \tag{9}$$

Looking at the terms in the integrand, we see that $N(p)$ is an even function of each of p^x, p^y and p^z. Since $p^t=mu^t=\frac{m^2}{m\sqrt{1-\beta^2}}=\frac{m^2}{\sqrt{m^2-p^2}}$, it too is an even function of each of p^x, p^y and p^z. If we consider an off-diagonal element of T^{ij} such as T^{yz}, we have

$$T^{yz}=\frac{1}{L^{3}}\int dp^{x}\int p^{y}dp^{y}\int p^{z}\frac{N(p)}{p^{t}}dp^{z} \tag{10}$$

The first integral over p^z gives zero, since we are integrating the product of an even function $N(p)/p^t$ and an odd function p^z over the range $-\infty$ to ∞. (We are also assuming, of course, that $N(p)\to 0$ as $p\to\infty$ which is reasonable, since we'd expect the number of particles with very large momenta to go to zero.)

The same argument applies to any off-diagonal element since any such element will integrate an odd function p^i multiplied by the even function $N(p)/p^t$. This argument applies also to off-diagonal elements where one of the indices is t, since the other index will be a spatial index and provide the odd function that multiplies $N(p)/p^t$.

If we look at the diagonal elements, however, we get the product of two even functions, as with T^{zz}:

$$T^{zz}=\frac{1}{L^{3}}\int dp^{x}\int dp^{y}\int \left(p^{z}\right)^{2}\frac{N(p)}{p^{t}}dp^{z} \tag{11}$$

and this will not, in general, be zero (unless no particles have a z component of momentum). Therefore, for the special case of a perfect fluid at rest, the stress-energy tensor is diagonal.

To interpret these elements we can use a clever little argument as follows. Look at a single particle that bounces elastically (that is, it doesn't lose any energy in a collision) off the walls of the cube. As seen in the fluid's frame, the time taken between bounces off, say, one of the faces perpendicular to the x axis is the total distance in the x direction it travels (which is twice the width of the cube, or $2L$) divided by its x speed:

$$\Delta t_{x}=\frac{2L}{v^{x}} \tag{12}$$

Since the collision is elastic, the particle rebounds from the wall with the same magnitude of v_x but the opposite direction, so its x momentum has been exactly reversed. This means that the wall imparted a change in x momentum of

$$\Delta p^{x}=2mv^{x} \tag{13}$$

The pressure is the average force per unit area, so for this single particle, the average force is the change in momentum per unit time, or

$$\Delta F^{x}=\frac{\Delta p^{x}}{\Delta t_{x}}=\frac{mv^{x}v^{x}}{L}=\frac{p^{x}v^{x}}{L} \tag{14}$$

The pressure is the force per unit area, so

$$\Delta P=\frac{\Delta F^{x}}{L^{2}}=\frac{p^{x}v^{x}}{L^{3}}=\frac{p^{x}\gamma m v^{x}}{\gamma m}=\frac{\left(p^{x}\right)^{2}}{p^{t}L^{3}} \tag{15}$$

(Note that v^x is the x component of the particle's velocity as measured in the fluid's frame, so the x component of its *four*-velocity is $\gamma v^x = u^x$.)

The total pressure is therefore the integral of this over all particles, which is

$$P=\int\int\int N\left(p\right)\frac{\left(p^{x}\right)^{2}}{p^{t}L^{3}}dp^{x}dp^{y}dp^{z}=T^{xx} \tag{16}$$

by comparison with 9.

There's nothing special about choosing the x direction in this calculation, so we would end up with the same result having chosen y or z. Thus

$$T^{xx}=T^{yy}=T^{zz}=P \tag{17}$$

For the remaining component T^{tt} we have

$$\begin{aligned} T^{tt} &= \int\int\int N\left(p\right)\frac{\left(p^{t}\right)^{2}}{p^{t}L^{3}}dp^{x}dp^{y}dp^{z} && (18)\\ &= \frac{1}{L^{3}}\int\int\int N\left(p\right)p^{t}dp^{x}dp^{y}dp^{z} && (19)\end{aligned}$$

Since p^t is the energy of a particle with four-momentum p^i, this integral represents the total energy density ρ of the fluid. We therefore get, for a perfect fluid in its rest frame:

$$T^{ij}=\left[\begin{array}{cccc} \rho & 0 & 0 & 0\\ 0 & P & 0 & 0\\ 0 & 0 & P & 0\\ 0 & 0 & 0 & P \end{array}\right] \tag{20}$$

PINGBACKS

Pingback: Stress-energy tensor at the centre of the sun
Pingback: Stress-energy tensor: relativistic perfect fluid
Pingback: Stress-energy tensor for perfect fluid: general coordinates
Pingback: Stress-energy tensor for a photon gas
Pingback: Metric tensor as a stress-energy tensor
Pingback: Stress-energy tensor in a local orthonormal frame
Pingback: Stress-energy tensor: negative pressure revisited
Pingback: Dominant energy condition

Pingback: Einstein equation in the Newtonian limit
Pingback: Einstein equation for an exponential metric