

STRESS-ENERGY TENSOR OF A SLOWLY ROTATING STAR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; Problem 20.3.

The general form of the stress-energy tensor for a perfect fluid is

$$(0.1) \quad T^{ij} = (\rho_0 + P_0) u^i u^j + P_0 g^{ij}$$

where ρ_0 is the energy density of the fluid and P_0 is the pressure, and u^i is the four-velocity of the fluid, all measured in the fluid's rest frame. Suppose we have a star with a spherically symmetric density $\rho(r)$ that rotates about the z axis with an angular velocity ω that is slow enough that the speed of the fluid making up the star at any point is non-relativistic, that is $v = \omega r \sin \theta \ll 1$ (where we're using the usual spherical coordinates r and θ).

We've seen that in a star like the sun, the pressure P_0 in the interior of the star is negligible relative to the energy density ρ_0 so we'll neglect P_0 in what follows. The tensor then becomes

$$(0.2) \quad T^{ij} \approx \rho_0 u^i u^j$$

The components of the four-velocity can be worked out as follows:

$$(0.3) \quad u^t = \gamma$$

$$(0.4) \quad = \frac{1}{\sqrt{1 - (v^x)^2 - (v^y)^2}}$$

$$(0.5) \quad u^x = \gamma v^x$$

$$(0.6) \quad = \frac{1}{\sqrt{1 - (v^x)^2 - (v^y)^2}} v^x$$

$$(0.7) \quad u^y = \frac{1}{\sqrt{1 - (v^x)^2 - (v^y)^2}} v^y$$

$$(0.8) \quad u^z = 0$$

The components of the velocity are

$$(0.9) \quad v^x = -(r \sin \theta) \omega \sin \phi$$

$$(0.10) \quad = -\omega y$$

$$(0.11) \quad v^y = (r \sin \theta) \omega \cos \phi$$

$$(0.12) \quad = \omega x$$

so we have

$$(0.13) \quad u^t = \gamma = \frac{1}{\sqrt{1 - \omega^2(x^2 + y^2)}}$$

$$(0.14) \quad u^x = -\gamma \omega y$$

$$(0.15) \quad u^y = \gamma \omega x$$

with the density as a function of x , y and z :

$$(0.16) \quad \rho = \rho(r) = \rho\left(\sqrt{x^2 + y^2 + z^2}\right)$$

We then get for T^{ij} :

$$(0.17) \quad T^{ij} = \rho \gamma^2 \begin{bmatrix} 1 & -\omega y & \omega x & 0 \\ -\omega y & \omega^2 y^2 & -\omega^2 xy & 0 \\ \omega x & -\omega^2 xy & \omega^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In the non-relativistic case, $\gamma \approx 1$ and $\omega^2 r^2 \ll \omega r \ll 1$ so we can approximate the tensor as

$$(0.18) \quad T^{ij} \approx \rho \begin{bmatrix} 1 & -\omega y & \omega x & 0 \\ -\omega y & 0 & 0 & 0 \\ \omega x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$