STRESS-ENERGY TENSOR OF A SLOWLY ROTATING STAR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; Problem 20.3.

The general form of the stress-energy tensor for a perfect fluid is

$$T^{ij}=(\rho_0+P_0)\,u^iu^j+P_0g^{ij} \tag{1}$$

where ρ_0 is the energy density of the fluid and P_0 is the pressure, and u^i is the four-velocity of the fluid, all measured in the fluid's rest frame. Suppose we have a star with a spherically symmetric density $\rho(r)$ that rotates about the z axis with an angular velocity ω that is slow enough that the speed of the fluid making up the star at any point is non-relativistic, that is $v=\omega r\sin\theta\ll1$ (where we're using the usual spherical coordinates r and θ).

We've seen that in a star like the sun, the pressure P_0 in the interior of the star is negligible relative to the energy density ρ_0 so we'll neglect P_0 in what follows. The tensor then becomes

$$T^{ij}\approx\rho_0u^iu^j \tag{2}$$

The components of the four-velocity can be worked out as follows:

$$\begin{aligned}
u^t &= \gamma && (3)\\
&= \frac{1}{\sqrt{1-(v^x)^2-(v^y)^2}} && (4)\\
u^x &= \gamma v^x && (5)\\
&= \frac{1}{\sqrt{1-(v^x)^2-(v^y)^2}}v^x && (6)\\
u^y &= \frac{1}{\sqrt{1-(v^x)^2-(v^y)^2}}v^y && (7)\\
u^z &= 0 && (8)
\end{aligned}$$

The components of the velocity are

$$\begin{aligned} v^x &= -(r\sin\theta)\,\omega\sin\phi && (9)\\ &= -\omega y && (10)\\ v^y &= (r\sin\theta)\,\omega\cos\phi && (11)\\ &= \omega x && (12) \end{aligned}$$

so we have

$$\begin{aligned} u^t &= \gamma = \frac{1}{\sqrt{1-\omega^2\left(x^2+y^2\right)}} && (13)\\ u^x &= -\gamma\omega y && (14)\\ u^y &= \gamma\omega x && (15) \end{aligned}$$

with the density as a function of x, y and z:

$$\rho = \rho(r) = \rho\left(\sqrt{x^2+y^2+z^2}\right) \tag{16}$$

We then get for T^{ij}:

$$T^{ij} = \rho\gamma^2 \begin{bmatrix} 1 & -\omega y & \omega x & 0 \\ -\omega y & \omega^2 y^2 & -\omega^2 xy & 0 \\ \omega x & -\omega^2 xy & \omega^2 x^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

In the non-relativistic case, $\gamma \approx 1$ and $\omega^2 r^2 \ll \omega r \ll 1$ so we can approximate the tensor as

$$T^{ij} \approx \rho \begin{bmatrix} 1 & -\omega y & \omega x & 0 \\ -\omega y & 0 & 0 & 0 \\ \omega x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$