STRESS-ENERGY TENSOR FOR A PHOTON GAS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; 4.

A special case of the stress-energy tensor for a perfect fluid at rest is that of an ideal gas of photons. The tensor in this case is

$$T^{ij}=\left[\begin{array}{cccc}\rho & 0 & 0 & 0\\ 0 & P & 0 & 0\\ 0 & 0 & P & 0\\ 0 & 0 & 0 & P\end{array}\right] \tag{1}$$

where ρ is the energy density and P is the pressure, both measured in the fluid's local inertial frame where the fluid as a whole is at rest. Since photons have no rest mass, we can go back to the original formula for the pressure and energy density

$$P = \frac{1}{V}\int\int\int N(p)\frac{(p^x)^2}{p^t}dp^x dp^y dp^z \tag{2}$$

$$\rho = \frac{1}{V}\int\int\int N(p)\,p^t dp^x dp^y dp^z \tag{3}$$

where p^i is the four-momentum and V is the volume of the container. For photons, the four-momentum is

$$\mathbf{p}=E\left[1,v^x,v^y,v^z\right] \tag{4}$$

where $(v^x)^2+(v^y)^2+(v^z)^2=1$. Thus

$$\rho = \frac{1}{V}\int\int\int N(p)\,E dp^x dp^y dp^z \tag{5}$$

$$P = \frac{1}{V}\int\int\int N(p)\,E\,(v^x)^2\,dp^x dp^y dp^z \tag{6}$$

For photons, the magnitude p of the spatial momentum is equal to the energy, so $p=E$. We can think of these integrals as integrating first over all combinations of spatial momentum that give a particular energy E and then integrating over E. For a fixed value of E, we can take it (and $N(p)$ which depends only on the magnitude p) outside the integral to get

$$P_E = \frac{N(p)E}{V}\int\int\int_E (v^x)^2\,dp^x dp^y dp^z \tag{7}$$

where the subscript E indicates we're integrating over the components of p such that $\sqrt{(p^x)^2+(p^y)^2+(p^z)^2} = E$. Since we're considering all possible directions for the photons' velocities, the integral is effectively just an average of $(v^x)^2$, which comes out to $1/3$. That is

$$P_E = \frac{N(p)E}{3V} \tag{8}$$

Since this is true for all energies, we get for the total pressure:

$$P = \frac{1}{3V}\int\int\int N(p)E\,dp^x dp^y dp^z = \frac{\rho}{3} \tag{9}$$

As there is nothing special about the x direction, we get the same result if we consider v^y and v^z so

$$T^{ij} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{3}\rho & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{bmatrix} \tag{10}$$