

STRESS-ENERGY TENSOR FOR A PHOTON GAS

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; 4.

A special case of the stress-energy tensor for a perfect fluid at rest is that of an ideal gas of photons. The tensor in this case is

$$(1) \quad T^{ij} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

where ρ is the energy density and P is the pressure, both measured in the fluid's local inertial frame where the fluid as a whole is at rest. Since photons have no rest mass, we can go back to the original formula for the pressure and energy density

$$(2) \quad P = \frac{1}{V} \int \int \int N(p) \frac{(p^x)^2}{p^t} dp^x dp^y dp^z$$

$$(3) \quad \rho = \frac{1}{V} \int \int \int N(p) p^t dp^x dp^y dp^z$$

where p^i is the four-momentum and V is the volume of the container. For photons, the four-momentum is

$$(4) \quad \mathbf{p} = E [1, v^x, v^y, v^z]$$

where $(v^x)^2 + (v^y)^2 + (v^z)^2 = 1$. Thus

$$(5) \quad \rho = \frac{1}{V} \int \int \int N(p) E dp^x dp^y dp^z$$

$$(6) \quad P = \frac{1}{V} \int \int \int N(p) E (v^x)^2 dp^x dp^y dp^z$$

For photons, the magnitude p of the spatial momentum is equal to the energy, so $p = E$. We can think of these integrals as integrating first over all

combinations of spatial momentum that give a particular energy E and then integrating over E . For a fixed value of E , we can take it (and $N(p)$ which depends only on the magnitude p) outside the integral to get

$$(7) \quad P_E = \frac{N(p)E}{V} \int \int \int_E (v^x)^2 dp^x dp^y dp^z$$

where the subscript E indicates we're integrating over the components of p such that $\sqrt{(p^x)^2 + (p^y)^2 + (p^z)^2} = E$. Since we're considering all possible directions for the photons' velocities, the integral is effectively just an average of $(v^x)^2$, which comes out to $1/3$. That is

$$(8) \quad P_E = \frac{N(p)E}{3V}$$

Since this is true for all energies, we get for the total pressure:

$$(9) \quad P = \frac{1}{3V} \int \int \int N(p) E dp^x dp^y dp^z = \frac{\rho}{3}$$

As there is nothing special about the x direction, we get the same result if we consider v^y and v^z so

$$(10) \quad T^{ij} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{3}\rho & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{bmatrix}$$