

## STRESS-ENERGY TENSOR FOR A PHOTON GAS

Link to: [physicspages home page](#).

To leave a comment or report an error, please use the [auxiliary blog](#).

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; 4.

A special case of the stress-energy tensor for a perfect fluid at rest is that of an ideal gas of photons. The tensor in this case is

$$T^{ij} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \quad (1)$$

where  $\rho$  is the energy density and  $P$  is the pressure, both measured in the fluid's local inertial frame where the fluid as a whole is at rest. Since photons have no rest mass, we can go back to the original formula for the pressure and energy density

$$P = \frac{1}{V} \int \int \int N(p) \frac{(p^x)^2}{p^t} dp^x dp^y dp^z \quad (2)$$

$$\rho = \frac{1}{V} \int \int \int N(p) p^t dp^x dp^y dp^z \quad (3)$$

where  $p^i$  is the four-momentum and  $V$  is the volume of the container. For photons, the four-momentum is

$$\mathbf{p} = E [1, v^x, v^y, v^z] \quad (4)$$

where  $(v^x)^2 + (v^y)^2 + (v^z)^2 = 1$ . Thus

$$\rho = \frac{1}{V} \int \int \int N(p) E dp^x dp^y dp^z \quad (5)$$

$$P = \frac{1}{V} \int \int \int N(p) E (v^x)^2 dp^x dp^y dp^z \quad (6)$$

For photons, the magnitude  $p$  of the spatial momentum is equal to the energy, so  $p = E$ . We can think of these integrals as integrating first over all combinations of spatial momentum that give a particular energy  $E$  and

then integrating over  $E$ . For a fixed value of  $E$ , we can take it (and  $N(p)$  which depends only on the magnitude  $p$ ) outside the integral to get

$$P_E = \frac{N(p)E}{V} \int \int \int_E (v^x)^2 dp^x dp^y dp^z \quad (7)$$

where the subscript  $E$  indicates we're integrating over the components of  $p$  such that  $\sqrt{(p^x)^2 + (p^y)^2 + (p^z)^2} = E$ . Since we're considering all possible directions for the photons' velocities, the integral is effectively just an average of  $(v^x)^2$ , which comes out to  $1/3$ . That is

$$P_E = \frac{N(p)E}{3V} \quad (8)$$

Since this is true for all energies, we get for the total pressure:

$$P = \frac{1}{3V} \int \int \int N(p) E dp^x dp^y dp^z = \frac{\rho}{3} \quad (9)$$

As there is nothing special about the  $x$  direction, we get the same result if we consider  $v^y$  and  $v^z$  so

$$T^{ij} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{3}\rho & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{bmatrix} \quad (10)$$