FORCE IN TERMS OF THE STRESS-ENERGY TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20, Problem 20.6.

A natural way of defining force as a four-vector is as the derivative with respect to proper time τ of the four-momentum, that is

$$F^i = \left(\frac{d\mathbf{p}}{d\tau}\right)^i \tag{1}$$

We can relate this to the stress-energy tensor by using the fact that T^{ij} is the rate of flow of p^i in direction j (when j is a spatial coordinate). If $i = t$, T^{tj} is the rate of flow of energy in the j direction, while if i is a spatial coordinate, T^{ij} is the rate of flow of that momentum component in the j direction. If we want to know the rate at which component i of four-momentum flows across a small planar patch of area A, we can take the unit normal to the area $\mathbf{n}$. If $\mathbf{n}$ is a four-vector then since it's a purely spatial direction, its time component is zero and we have

$$\mathbf{n} = [0, n^x, n^y, n^z] \tag{2}$$

Just as the component of an ordinary vector along a particular direction is given by the scalar product, we can think of one row in the stress-energy tensor as an analogue of a four-vector. For example, if we take the second row we get the components

$$\mathbf{T}_x = \left[T^{xt}, T^{xx}, T^{xy}, T^{xz}\right] \tag{3}$$

The first component $T^{xt} = T^{tx}$ is the rate of flow of energy in the x direction, while the last three components give the rate of flow of x momentum as a 3-d vector. The rate of flow of x momentum in the $\mathbf{n}$ direction is then

$$\begin{aligned}\left(\frac{d\mathbf{p}}{d\tau}\right)^x &= A\mathbf{n}\cdot\mathbf{T}_x & (4)\\ &= A g_{ij} n^i T^{xj} & (5)\end{aligned}$$

where we've multiplied by A since the components T^{xj} give the rate of flow per unit area. Notice this formula works because $n^t = 0$, so only the three components of spatial momentum contribute to this product. The same formula applies to the y and z components as well.

What about $\left(\frac{d\mathbf{p}}{d\tau}\right)^t$? If we apply the same idea, we have

$$\mathbf{T}_t = \left[T^{tt}, T^{tx}, T^{ty}, T^{tz}\right] \tag{6}$$

and

$$\left(\frac{d\mathbf{p}}{d\tau}\right)^t = A g_{ij} n^i T^{tj} \tag{7}$$

This is the rate of energy flow along the direction **n**. Thus the general formula for the rate of four-momentum flow is

$$\left(\frac{d\mathbf{p}}{d\tau}\right)^i = A g_{kj} n^k T^{ij} \tag{8}$$

In the general case where the fluid's four-velocity is u^i, the stress-energy tensor is

$$T^{ij} = (\rho_0 + P_0)\, u^i u^j + P_0 g^{ij} \tag{9}$$

where ρ_0 is the energy density of the fluid and P_0 is the pressure, both measured in the fluid's rest frame. The momentum flow in the fluid's rest frame is

$$\left(\frac{d\mathbf{p}}{d\tau}\right)^i = A g_{kj} n^k \left[(\rho_0 + P_0)\, u^i u^j + P_0 g^{ij}\right] \tag{10}$$

However, in the fluid's rest frame, the fluid's own four-velocity is $u^i = [1,0,0,0]$ so combined with 2 we have

$$\mathbf{u} \cdot \mathbf{n} = g_{kj} u^j n^k = 0 \tag{11}$$

Since this is a scalar, it has the same value in all coordinate systems, so the first term in 10 is zero in every coordinate system, so we get

$$\begin{aligned} \left(\frac{d\mathbf{p}}{d\tau}\right)^i &= A g_{kj} n^k P_0 g^{ij} && (12) \\ &= A \delta^i_k n^k P_0 && (13) \\ &= A n^i P_0 && (14) \end{aligned}$$

If now we place a real wall at the patch A, and this wall absorbs all the momentum that flows into it, then $\frac{d\mathbf{p}}{d\tau}$ is the force on that wall. The magnitude of the force is

$$
\begin{aligned}
F &= \sqrt{\frac{d\mathbf{p}}{d\tau} \cdot \frac{d\mathbf{p}}{d\tau}} && (15)\\
&= AP_0\sqrt{g_{ij}n^i n^j} && (16)\\
&= AP_0 && (17)
\end{aligned}
$$

where the result follows because $\mathbf{n}$ is a unit vector. Again, this result is a scalar so it applies in all coordinate systems.