

FORCE IN TERMS OF THE STRESS-ENERGY TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20, Problem 20.6.

A natural way of defining force as a four-vector is as the derivative with respect to proper time τ of the four-momentum, that is

$$(0.1) \quad F^i = \left(\frac{d\mathbf{p}}{d\tau} \right)^i$$

We can relate this to the stress-energy tensor by using the fact that T^{ij} is the rate of flow of p^i in direction j (when j is a spatial coordinate). If $i = t$, T^{tj} is the rate of flow of energy in the j direction, while if i is a spatial coordinate, T^{ij} is the rate of flow of that momentum component in the j direction. If we want to know the rate at which component i of four-momentum flows across a small planar patch of area A , we can take the unit normal to the area \mathbf{n} . If \mathbf{n} is a four-vector then since it's a purely spatial direction, its time component is zero and we have

$$(0.2) \quad \mathbf{n} = [0, n^x, n^y, n^z]$$

Just as the component of an ordinary vector along a particular direction is given by the scalar product, we can think of one row in the stress-energy tensor as an analogue of a four-vector. For example, if we take the second row we get the components

$$(0.3) \quad \mathbf{T}_x = [T^{xt}, T^{xx}, T^{xy}, T^{xz}]$$

The first component $T^{xt} = T^{tx}$ is the rate of flow of energy in the x direction, while the last three components give the rate of flow of x momentum as a 3-d vector. The rate of flow of x momentum in the \mathbf{n} direction is then

$$(0.4) \quad \left(\frac{d\mathbf{p}}{d\tau} \right)^x = \mathbf{An} \cdot \mathbf{T}_x$$

$$(0.5) \quad = \sum_1 A g_{ij} n^i T^{xj}$$

where we've multiplied by A since the components T^{xj} give the rate of flow per unit area. Notice this formula works because $n^t = 0$, so only the three components of spatial momentum contribute to this product. The same formula applies to the y and z components as well.

What about $\left(\frac{d\mathbf{p}}{d\tau}\right)^t$? If we apply the same idea, we have

$$(0.6) \quad \mathbf{T}_t = [T^{tt}, T^{tx}, T^{ty}, T^{tz}]$$

and

$$(0.7) \quad \left(\frac{d\mathbf{p}}{d\tau}\right)^t = Ag_{ij}n^i T^{tj}$$

This is the rate of energy flow along the direction \mathbf{n} . Thus the general formula for the rate of four-momentum flow is

$$(0.8) \quad \left(\frac{d\mathbf{p}}{d\tau}\right)^i = Ag_{kj}n^k T^{ij}$$

In the general case where the fluid's four-velocity is u^i , the stress-energy tensor is

$$(0.9) \quad T^{ij} = (\rho_0 + P_0)u^i u^j + P_0 g^{ij}$$

where ρ_0 is the energy density of the fluid and P_0 is the pressure, both measured in the fluid's rest frame. The momentum flow in the fluid's rest frame is

$$(0.10) \quad \left(\frac{d\mathbf{p}}{d\tau}\right)^i = Ag_{kj}n^k [(\rho_0 + P_0)u^i u^j + P_0 g^{ij}]$$

However, in the fluid's rest frame, the fluid's own four-velocity is $u^i = [1, 0, 0, 0]$ so combined with 0.9 we have

$$(0.11) \quad \mathbf{u} \cdot \mathbf{n} = g_{kj}u^j n^k = 0$$

Since this is a scalar, it has the same value in all coordinate systems, so the first term in 0.10 is zero in every coordinate system, so we get

$$(0.12) \quad \left(\frac{d\mathbf{p}}{d\tau} \right)^i = A g_{kj} n^k P_0 g^{ij}$$

$$(0.13) \quad = A \delta_k^i n^k P_0$$

$$(0.14) \quad = A n^i P_0$$

If now we place a real wall at the patch A , and this wall absorbs all the momentum that flows into it, then $\frac{d\mathbf{p}}{d\tau}$ is the force on that wall. The magnitude of the force is

$$(0.15) \quad F = \sqrt{\frac{d\mathbf{p}}{d\tau} \cdot \frac{d\mathbf{p}}{d\tau}}$$

$$(0.16) \quad = A P_0 \sqrt{g_{ij} n^i n^j}$$

$$(0.17) \quad = A P_0$$

where the result follows because \mathbf{n} is a unit vector. Again, this result is a scalar so it applies in all coordinate systems.