

FORCE IN TERMS OF THE STRESS-ENERGY TENSOR

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20, Problem 20.6.

A natural way of defining force as a four-vector is as the derivative with respect to proper time τ of the four-momentum, that is

$$F^i = \left(\frac{d\mathbf{p}}{d\tau} \right)^i \quad (1)$$

We can relate this to the stress-energy tensor by using the fact that T^{ij} is the rate of flow of p^i in direction j (when j is a spatial coordinate). If $i = t$, T^{tj} is the rate of flow of energy in the j direction, while if i is a spatial coordinate, T^{ij} is the rate of flow of that momentum component in the j direction. If we want to know the rate at which component i of four-momentum flows across a small planar patch of area A , we can take the unit normal to the area \mathbf{n} . If \mathbf{n} is a four-vector then since it's a purely spatial direction, its time component is zero and we have

$$\mathbf{n} = [0, n^x, n^y, n^z] \quad (2)$$

Just as the component of an ordinary vector along a particular direction is given by the scalar product, we can think of one row in the stress-energy tensor as an analogue of a four-vector. For example, if we take the second row we get the components

$$\mathbf{T}_x = [T^{xt}, T^{xx}, T^{xy}, T^{xz}] \quad (3)$$

The first component $T^{xt} = T^{tx}$ is the rate of flow of energy in the x direction, while the last three components give the rate of flow of x momentum as a 3-d vector. The rate of flow of x momentum in the \mathbf{n} direction is then

$$\left(\frac{d\mathbf{p}}{d\tau} \right)^x = \mathbf{An} \cdot \mathbf{T}_x \quad (4)$$

$$= Ag_{ij}n^i T^{xj} \quad (5)$$

where we've multiplied by A since the components T^{xj} give the rate of flow per unit area. Notice this formula works because $n^t = 0$, so only the three

components of spatial momentum contribute to this product. The same formula applies to the y and z components as well.

What about $\left(\frac{d\mathbf{p}}{d\tau}\right)^t$? If we apply the same idea, we have

$$\mathbf{T}_t = [T^{tt}, T^{tx}, T^{ty}, T^{tz}] \quad (6)$$

and

$$\left(\frac{d\mathbf{p}}{d\tau}\right)^t = Ag_{ij}n^i T^{tj} \quad (7)$$

This is the rate of energy flow along the direction \mathbf{n} . Thus the general formula for the rate of four-momentum flow is

$$\left(\frac{d\mathbf{p}}{d\tau}\right)^i = Ag_{kj}n^k T^{ij} \quad (8)$$

In the general case where the fluid's four-velocity is u^i , the stress-energy tensor is

$$T^{ij} = (\rho_0 + P_0)u^i u^j + P_0 g^{ij} \quad (9)$$

where ρ_0 is the energy density of the fluid and P_0 is the pressure, both measured in the fluid's rest frame. The momentum flow in the fluid's rest frame is

$$\left(\frac{d\mathbf{p}}{d\tau}\right)^i = Ag_{kj}n^k [(\rho_0 + P_0)u^i u^j + P_0 g^{ij}] \quad (10)$$

However, in the fluid's rest frame, the fluid's own four-velocity is $u^i = [1, 0, 0, 0]$ so combined with 2 we have

$$\mathbf{u} \cdot \mathbf{n} = g_{kj}u^j n^k = 0 \quad (11)$$

Since this is a scalar, it has the same value in all coordinate systems, so the first term in 10 is zero in every coordinate system, so we get

$$\left(\frac{d\mathbf{p}}{d\tau}\right)^i = Ag_{kj}n^k P_0 g^{ij} \quad (12)$$

$$= A\delta_k^i n^k P_0 \quad (13)$$

$$= An^i P_0 \quad (14)$$

If now we place a real wall at the patch A , and this wall absorbs all the momentum that flows into it, then $\frac{d\mathbf{p}}{d\tau}$ is the force on that wall. The magnitude of the force is

$$F = \sqrt{\frac{d\mathbf{p}}{d\tau} \cdot \frac{d\mathbf{p}}{d\tau}} \quad (15)$$

$$= AP_0 \sqrt{g_{ij} n^i n^j} \quad (16)$$

$$= AP_0 \quad (17)$$

where the result follows because \mathbf{n} is a unit vector. Again, this result is a scalar so it applies in all coordinate systems.