STRESS-ENERGY TENSOR IN A LOCAL ORTHONORMAL FRAME

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; Problem 20.8.

Earlier we saw that it's possible to define a local orthogonal frame (LOF) at any event in spacetime, with that frame embedded in a global system. Our earlier example defined a locally flat frame embedded within the Schwarzschild metric, but we can also use it to compare two inertial frames, with one frame moving with a four-velocity $\mathbf{u}_{obs}$ relative to the other.

As before, we define the observer's local basis vectors as $\mathbf{o}_i$ with coordinates in the observer's frame:

$$\mathbf{o}_t = [1,0,0,0] \tag{1}$$
$$\mathbf{o}_x = [0,1,0,0] \tag{2}$$
$$\mathbf{o}_y = [0,0,1,0] \tag{3}$$
$$\mathbf{o}_z = [0,0,0,1] \tag{4}$$

To find the components of a four-vector A^i in the observer's frame, we can work out the scalar product in any frame and, since a scalar is invariant, it will have the same value in all frames. That is

$$A^i_{obs} = \eta^{ij} g_{km} \left(\mathbf{o}_j\right)^k A^m \tag{5}$$

where g_{km} is the metric in the global frame and η^{ij} is the flat metric in the observer's frame.

For a second-rank tensor that is the product of two four-vectors, that is

$$C^{ij} = A^i B^j \tag{6}$$

we can just apply the transformation to each four-vector separately, so we get

$$C^{ip}_{obs} = \eta^{ij} g_{km} \left(\mathbf{o}_j\right)^k A^m \eta^{pq} g_{rs} \left(\mathbf{o}_q\right)^r B^s \tag{7}$$

$$= \eta^{ij} g_{km} \left(\mathbf{o}_j\right)^k \eta^{pq} g_{rs} \left(\mathbf{o}_q\right)^r C^{ms} \tag{8}$$

Although not all second-rank tensors are the product of two four-vectors, this seems a reasonable definition of the transformation.

Suppose that we now have a perfect fluid at rest in its own frame and an observer that moves with four-velocity $\mathbf{u}_{obs}$ relative to the fluid's frame. In this case, the global frame is the rest frame of the fluid. In the observer's own local frame, because his metric is flat and the observer is not moving relative to himself, $\mathbf{u} = [1,0,0,0]$. That is, in the local frame, $\mathbf{u} = \mathbf{o}_t$. Therefore, $\mathbf{u}_{obs}$ in the global frame is the transformed version of $\mathbf{o}_t$ so in the global frame

$$\mathbf{o}_t = \mathbf{u}_{obs} \tag{9}$$

$$= [\gamma, \gamma v^x, \gamma v^y, \gamma v^z] \tag{10}$$

In the fluid's frame, the stress-energy tensor is

$$T^{ij} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \tag{11}$$

We can find the energy density seen by the moving observer by applying 8 with $C^{ij} = T^{ij}$. Since both frames are inertial, both metrics are the flat space metric, that is, $g_{ij} = \eta_{ij}$. We thus get

$$T^{tt}_{obs} = \eta^{tj} \eta_{km} \left(\mathbf{o}_j\right)^k \eta^{tq} \eta_{rs} \left(\mathbf{o}_q\right)^r T^{ms} \tag{12}$$

Because η_{ij} and T^{ms} are diagonal, the only non-zero terms in the sum are with $j = t$, $q = t$, $k = m$, $r = s$ and $m = s$, so we get

$$
\begin{aligned}
T_{obs}^{tt} &= \eta_{km}\left(\mathbf{o}_t\right)^k \eta_{rs}\left(\mathbf{o}_t\right)^r T^{ms} && (13)\\
&= \left(\mathbf{o}_t\right)^t \left(\mathbf{o}_t\right)^t T^{tt} + \sum_{i=x,y,z} \left(\mathbf{o}_t\right)^i \left(\mathbf{o}_t\right)^i T^{ii} && (14)\\
&= \rho\gamma^2 + P\gamma^2 v^2 && (15)\\
&= \rho\gamma^2 + P\gamma^2\left(1-1+v^2\right) && (16)\\
&= (\rho+P)\,\gamma^2 + P\gamma^2\left(v^2-1\right) && (17)\\
&= (\rho+P)\,\gamma^2 - P && (18)
\end{aligned}
$$

The general form for the stress-energy tensor is

$$T^{ij} = (\rho+P)\,u^i u^j + P g^{ij} \tag{19}$$

so the tt component is

$$
\begin{aligned}
T^{tt} &= (\rho+P)\,u^t u^t + \eta^{tt} P && (20)\\
&= (\rho+P)\,\gamma^2 - P && (21)
\end{aligned}
$$

so we get the same answer.

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