DOMINANT ENERGY CONDITION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; Problem 20.10.

The *dominant energy condition* (DEC) states that if a^i is any four-vector that is *causal*, that is, it satisfies the conditions

$$\mathbf{a}\cdot\mathbf{a} \leq 0 \tag{1}$$

$$a^t > 0 \tag{2}$$

then we require the stress-energy tensor T^{ij} to satisfy the condition that if

$$b^i = -T^{ij}g_{jk}a^k \tag{3}$$

then $\mathbf{b}$ is also a causal four-vector. The causal condition is just a way of saying that a four-vector is either timelike (if $\mathbf{a}\cdot\mathbf{a}<0$) or lightlike (if $\mathbf{a}\cdot\mathbf{a}=0$). The DEC is a condition on the stress-energy tensor which amounts to saying that taking the scalar product of one of its rows or columns with a causal vector cannot produce a non-causal (spacelike) vector. Physically, this says that nothing can move faster than light. Note that it's not a property that is automatically true of any stress-energy tensor; rather it is a condition imposed on the tensor to make it physically realistic.

We can use the DEC to show that the momentum density of a perfect fluid is always causal. The tensor in the fluid's rest frame is

$$T^{ij}=\left[\begin{array}{cccc} \rho & 0 & 0 & 0\\ 0 & P & 0 & 0\\ 0 & 0 & P & 0\\ 0 & 0 & 0 & P \end{array}\right] \tag{4}$$

The momentum density is defined as the first row (or column) of the tensor:

$$\pi^i \equiv T^{ti} \tag{5}$$

In the rest frame,

$$\pi^i = [\rho, 0, 0, 0] \tag{6}$$
$$\pi \cdot \pi = -\rho^2 \leq 0 \tag{7}$$
$$\pi^t = \rho > 0 \tag{8}$$

so π is causal in this frame. In a local orthonormal frame (LOF) the tensor's components are

$$T^{ip}_{obs} = \eta^{ij}\eta_{km}\left(\mathbf{o}_j\right)^k \eta^{pq}\eta_{rs}\left(\mathbf{o}_q\right)^r T^{ms} \tag{9}$$

where $\mathbf{o}_i$ are the orthonormal basis vectors in the LOF. If we plug in the definition 5 we get

$$\pi^p_{obs} = T^{tp}_{obs} \tag{10}$$
$$= \eta^{tj}\eta_{km}\left(\mathbf{o}_j\right)^k \eta^{pq}\eta_{rs}\left(\mathbf{o}_q\right)^r T^{ms} \tag{11}$$
$$= \left[-T^{ms}\eta_{km}\left(\mathbf{o}_t\right)^k\right]\eta^{pq}\eta_{rs}\left(\mathbf{o}_q\right)^r \tag{12}$$

where we got the last line by using the fact that η^{ij} is diagonal and $\eta^{tt} = -1$. The term in square brackets looks like 3, as long as $\mathbf{o}_t$ is a causal vector. However, this vector is just the observer's four-velocity $\mathbf{u}_{obs}$ measured in the fluid's frame, so

$$\mathbf{u}_{obs} \cdot \mathbf{u}_{obs} = -1 \tag{13}$$
$$\mathbf{u}^t_{obs} = \gamma > 0 \tag{14}$$

Thus $\mathbf{o}_t$ is indeed causal, so we can invoke the DEC to say that if we define a vector B^s by

$$B^s \equiv -T^{ms}\eta_{km}\left(\mathbf{o}_t\right)^k \tag{15}$$

then B^s must be causal. We then get

$$\pi^p_{obs} = B^s \eta^{pq} \eta_{rs} \left(\mathbf{o}_q\right)^r \tag{16}$$
$$= \eta^{pq} \left[B^s \eta_{rs} \left(\mathbf{o}_q\right)^r\right] \tag{17}$$
$$= \eta^{pq} B_{obs,q} \tag{18}$$
$$= B^p_{obs} \tag{19}$$

With this definition, we can calculate

$$\pi_{obs} \cdot \pi_{obs} = \mathbf{B}_{obs} \cdot \mathbf{B}_{obs} \tag{20}$$

However, we know that **B** is causal because that's how we defined it in 15 and since its magnitude is a scalar, it is the same in all coordinate systems, so we must have $\pi_{obs} \cdot \pi_{obs} = \mathbf{B}_{obs} \cdot \mathbf{B}_{obs} \leq 0$. As for showing that $\pi^t_{obs} > 0$, we can observe that

$$\pi^t_{obs} = \eta^{tq} \left[B^s \eta_{rs} \left(\mathbf{o}_q\right)^r\right] \tag{21}$$
$$= -B^s \eta_{rs} \left(\mathbf{o}_t\right)^r \tag{22}$$

Since $\mathbf{o}_t = \gamma[1, v^x, v^y, v^z]$ we see that π^t_{obs} is the Lorentz transformation of the causal vector B^s and a Lorentz transformation never changes the space-time nature of a four-vector (that is, timelike remains timelike, etc) so since **B** is causal, $B^t > 0$ and therefore $\pi^t_{obs} > 0$ as well. This condition is known as the *weak energy condition* or WEC.

Another property of the stress-energy tensor that can be derived from the DEC is as follows. In the rest frame of a perfect fluid, 4 holds, so the DEC condition 3 says, for some arbitrary causal vector **a** we get another causal vector **b**:

$$b^i = -T^{ij} g_{jk} a^k \tag{23}$$
$$b^t = \rho a^t \tag{24}$$
$$b^m = -P a^m \text{ for } m = x, y, z \tag{25}$$

Since **b** is causal, we must have for *all* choices of **a**:

$$\mathbf{b} \cdot \mathbf{b} = -\rho^2 \left(a^t\right)^2 + P^2 \sum_{m=x,y,z} \left(a^m\right)^2 \leq 0 \tag{26}$$

Because **a** is causal, we have

$$\left(a^{t}\right)^{2} \geq \sum_{m=x,y,z}\left(a^{m}\right)^{2} \tag{27}$$

The constraint on ρ and P in 26 comes in the case of equality in 27, in which case we have

$$-\rho^{2}+P^{2} \leq 0 \tag{28}$$

and since $\rho > 0$ this amounts to

$$\rho \geq |P| \tag{29}$$

Finally, we can revisit the case of the stress-energy tensor that gives negative pressure, $T^{ij} = -\Lambda g^{ij}$, where Λ is a positive scalar. In this case, if we apply the DEC to some causal vector **a** we get

$$b^{i} = -T^{ij} g_{jk} a^{k} \tag{30}$$

$$= \Lambda g^{ij} g_{jk} a^{k} \tag{31}$$

$$= \Lambda a^{i} \tag{32}$$

Since **b** is a positive scalar multiplied by a causal vector, it too must be causal, so this stress-energy tensor satisfies the DEC.

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