

DOMINANT ENERGY CONDITION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; Problem 20.10.

The *dominant energy condition* (DEC) states that if a^i is any four-vector that is *causal*, that is, it satisfies the conditions

$$\mathbf{a} \cdot \mathbf{a} \leq 0 \tag{1}$$

$$a^t > 0 \tag{2}$$

then we require the stress-energy tensor T^{ij} to satisfy the condition that if

$$b^i = -T^{ij}g_{jk}a^k \tag{3}$$

then \mathbf{b} is also a causal four-vector. The causal condition is just a way of saying that a four-vector is either timelike (if $\mathbf{a} \cdot \mathbf{a} < 0$) or lightlike (if $\mathbf{a} \cdot \mathbf{a} = 0$). The DEC is a condition on the stress-energy tensor which amounts to saying that taking the scalar product of one of its rows or columns with a causal vector cannot produce a non-causal (spacelike) vector. Physically, this says that nothing can move faster than light. Note that it's not a property that is automatically true of any stress-energy tensor; rather it is a condition imposed on the tensor to make it physically realistic.

We can use the DEC to show that the momentum density of a perfect fluid is always causal. The tensor in the fluid's rest frame is

$$T^{ij} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix} \tag{4}$$

The momentum density is defined as the first row (or column) of the tensor:

$$\pi^i \equiv T^{ti} \tag{5}$$

In the rest frame,

$$\boldsymbol{\pi}^i = [\rho, 0, 0, 0] \quad (6)$$

$$\boldsymbol{\pi} \cdot \boldsymbol{\pi} = -\rho^2 \leq 0 \quad (7)$$

$$\boldsymbol{\pi}^t = \rho > 0 \quad (8)$$

so $\boldsymbol{\pi}$ is causal in this frame. In a local orthonormal frame (LOF) the tensor's components are

$$T_{obs}^{ip} = \eta^{ij} \eta_{km} (\mathbf{o}_j)^k \eta^{pq} \eta_{rs} (\mathbf{o}_q)^r T^{ms} \quad (9)$$

where \mathbf{o}_i are the orthonormal basis vectors in the LOF. If we plug in the definition 5 we get

$$\boldsymbol{\pi}_{obs}^p = T_{obs}^{tp} \quad (10)$$

$$= \eta^{tj} \eta_{km} (\mathbf{o}_j)^k \eta^{pq} \eta_{rs} (\mathbf{o}_q)^r T^{ms} \quad (11)$$

$$= \left[-T^{ms} \eta_{km} (\mathbf{o}_t)^k \right] \eta^{pq} \eta_{rs} (\mathbf{o}_q)^r \quad (12)$$

where we got the last line by using the fact that η^{ij} is diagonal and $\eta^{tt} = -1$. The term in square brackets looks like 3, as long as \mathbf{o}_t is a causal vector. However, this vector is just the observer's four-velocity \mathbf{u}_{obs} measured in the fluid's frame, so

$$\mathbf{u}_{obs} \cdot \mathbf{u}_{obs} = -1 \quad (13)$$

$$\mathbf{u}_{obs}^t = \gamma > 0 \quad (14)$$

Thus \mathbf{o}_t is indeed causal, so we can invoke the DEC to say that if we define a vector B^s by

$$B^s \equiv -T^{ms} \eta_{km} (\mathbf{o}_t)^k \quad (15)$$

then B^s must be causal. We then get

$$\boldsymbol{\pi}_{obs}^p = B^s \eta^{pq} \eta_{rs} (\mathbf{o}_q)^r \quad (16)$$

$$= \eta^{pq} [B^s \eta_{rs} (\mathbf{o}_q)^r] \quad (17)$$

$$= \eta^{pq} B_{obs,q} \quad (18)$$

$$= B_{obs}^p \quad (19)$$

With this definition, we can calculate

$$\boldsymbol{\pi}_{obs} \cdot \boldsymbol{\pi}_{obs} = \mathbf{B}_{obs} \cdot \mathbf{B}_{obs} \quad (20)$$

However, we know that \mathbf{B} is causal because that's how we defined it in 15 and since its magnitude is a scalar, it is the same in all coordinate systems, so we must have $\boldsymbol{\pi}_{obs} \cdot \boldsymbol{\pi}_{obs} = \mathbf{B}_{obs} \cdot \mathbf{B}_{obs} \leq 0$. As for showing that $\pi_{obs}^t > 0$, we can observe that

$$\pi_{obs}^t = \eta^{tq} [B^s \eta_{rs} (\mathbf{o}_q)^r] \quad (21)$$

$$= -B^s \eta_{rs} (\mathbf{o}_t)^r \quad (22)$$

Since $\mathbf{o}_t = \gamma[1, v^x, v^y, v^z]$ we see that π_{obs}^t is the Lorentz transformation of the causal vector B^s and a Lorentz transformation never changes the space-time nature of a four-vector (that is, timelike remains timelike, etc) so since \mathbf{B} is causal, $B^t > 0$ and therefore $\pi_{obs}^t > 0$ as well. This condition is known as the *weak energy condition* or WEC.

Another property of the stress-energy tensor that can be derived from the DEC is as follows. In the rest frame of a perfect fluid, 4 holds, so the DEC condition 3 says, for some arbitrary causal vector \mathbf{a} we get another causal vector \mathbf{b} :

$$b^i = -T^{ij} g_{jk} a^k \quad (23)$$

$$b^t = \rho a^t \quad (24)$$

$$b^m = -P a^m \text{ for } m = x, y, z \quad (25)$$

Since \mathbf{b} is causal, we must have for *all* choices of \mathbf{a} :

$$\mathbf{b} \cdot \mathbf{b} = -\rho^2 (a^t)^2 + P^2 \sum_{m=x,y,z} (a^m)^2 \leq 0 \quad (26)$$

Because \mathbf{a} is causal, we have

$$(a^t)^2 \geq \sum_{m=x,y,z} (a^m)^2 \quad (27)$$

The constraint on ρ and P in 26 comes in the case of equality in 27, in which case we have

$$-\rho^2 + P^2 \leq 0 \quad (28)$$

and since $\rho > 0$ this amounts to

$$\rho \geq |P| \quad (29)$$

Finally, we can revisit the case of the stress-energy tensor that gives negative pressure, $T^{ij} = -\Lambda g^{ij}$, where Λ is a positive scalar. In this case, if we apply the DEC to some causal vector \mathbf{a} we get

$$b^i = -T^{ij} g_{jk} a^k \quad (30)$$

$$= \Lambda g^{ij} g_{jk} a^k \quad (31)$$

$$= \Lambda a^i \quad (32)$$

Since \mathbf{b} is a positive scalar multiplied by a causal vector, it too must be causal, so this stress-energy tensor satisfies the DEC.

PINGBACKS

Pingback: Vacuum stress-energy and the cosmological constant