

DOMINANT ENERGY CONDITION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 20; Problem 20.10.

The *dominant energy condition* (DEC) states that if a^i is any four-vector that is *causal*, that is, it satisfies the conditions

$$(0.1) \quad \mathbf{a} \cdot \mathbf{a} \leq 0$$

$$(0.2) \quad a^t > 0$$

then we require the stress-energy tensor T^{ij} to satisfy the condition that if

$$(0.3) \quad b^i = -T^{ij}g_{jk}a^k$$

then \mathbf{b} is also a causal four-vector. The causal condition is just a way of saying that a four-vector is either timelike (if $\mathbf{a} \cdot \mathbf{a} < 0$) or lightlike (if $\mathbf{a} \cdot \mathbf{a} = 0$). The DEC is a condition on the stress-energy tensor which amounts to saying that taking the scalar product of one of its rows or columns with a causal vector cannot produce a non-causal (spacelike) vector. Physically, this says that nothing can move faster than light. Note that it's not a property that is automatically true of any stress-energy tensor; rather it is a condition imposed on the tensor to make it physically realistic.

We can use the DEC to show that the momentum density of a perfect fluid is always causal. The tensor in the fluid's rest frame is

$$(0.4) \quad T^{ij} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

The momentum density is defined as the first row (or column) of the tensor:

$$(0.5) \quad \pi^i \equiv T^{ti}$$

In the rest frame,

$$(0.6) \quad \pi^i = [\rho, 0, 0, 0]$$

$$(0.7) \quad \pi \cdot \pi = -\rho^2 \leq 0$$

$$(0.8) \quad \pi^t = \rho > 0$$

so π is causal in this frame. In a local orthonormal frame (LOF) the tensor's components are

$$(0.9) \quad T_{obs}^{ip} = \eta^{ij} \eta_{km} (\mathbf{o}_j)^k \eta^{pq} \eta_{rs} (\mathbf{o}_q)^r T^{ms}$$

where \mathbf{o}_i are the orthonormal basis vectors in the LOF. If we plug in the definition 0.5 we get

$$(0.10) \quad \pi_{obs}^p = T_{obs}^{tp}$$

$$(0.11) \quad = \eta^{tj} \eta_{km} (\mathbf{o}_j)^k \eta^{pq} \eta_{rs} (\mathbf{o}_q)^r T^{ms}$$

$$(0.12) \quad = \left[-T^{ms} \eta_{km} (\mathbf{o}_t)^k \right] \eta^{pq} \eta_{rs} (\mathbf{o}_q)^r$$

where we got the last line by using the fact that η^{ij} is diagonal and $\eta^{tt} = -1$. The term in square brackets looks like 0.3, as long as \mathbf{o}_t is a causal vector. However, this vector is just the observer's four-velocity \mathbf{u}_{obs} measured in the fluid's frame, so

$$(0.13) \quad \mathbf{u}_{obs} \cdot \mathbf{u}_{obs} = -1$$

$$(0.14) \quad \mathbf{u}_{obs}^t = \gamma > 0$$

Thus \mathbf{o}_t is indeed causal, so we can invoke the DEC to say that if we define a vector B^s by

$$(0.15) \quad B^s \equiv -T^{ms} \eta_{km} (\mathbf{o}_t)^k$$

then B^s must be causal. We then get

$$\begin{aligned}
(0.16) \quad \pi_{obs}^p &= B^s \eta^{pq} \eta_{rs} (\mathbf{o}_q)^r \\
(0.17) \quad &= \eta^{pq} [B^s \eta_{rs} (\mathbf{o}_q)^r] \\
(0.18) \quad &= \eta^{pq} B_{obs,q} \\
(0.19) \quad &= B_{obs}^p
\end{aligned}$$

With this definition, we can calculate

$$(0.20) \quad \pi_{obs} \cdot \pi_{obs} = \mathbf{B}_{obs} \cdot \mathbf{B}_{obs}$$

However, we know that \mathbf{B} is causal because that's how we defined it in 0.15 and since its magnitude is a scalar, it is the same in all coordinate systems, so we must have $\pi_{obs} \cdot \pi_{obs} = \mathbf{B}_{obs} \cdot \mathbf{B}_{obs} \leq 0$. As for showing that $\pi_{obs}^t > 0$, we can observe that

$$(0.21) \quad \pi_{obs}^t = \eta^{tq} [B^s \eta_{rs} (\mathbf{o}_q)^r]$$

$$(0.22) \quad = -B^s \eta_{rs} (\mathbf{o}_t)^r$$

Since $\mathbf{o}_t = \gamma[1, v^x, v^y, v^z]$ we see that π_{obs}^t is the Lorentz transformation of the causal vector B^s and a Lorentz transformation never changes the space-time nature of a four-vector (that is, timelike remains timelike, etc) so since \mathbf{B} is causal, $B^t > 0$ and therefore $\pi_{obs}^t > 0$ as well. This condition is known as the *weak energy condition* or WEC.

Another property of the stress-energy tensor that can be derived from the DEC is as follows. In the rest frame of a perfect fluid, 0.4 holds, so the DEC condition 0.3 says, for some arbitrary causal vector \mathbf{a} we get another causal vector \mathbf{b} :

$$(0.23) \quad b^i = -T^{ij} g_{jk} a^k$$

$$(0.24) \quad b^t = \rho a^t$$

$$(0.25) \quad b^m = -P a^m \text{ for } m = x, y, z$$

Since \mathbf{b} is causal, we must have for *all* choices of \mathbf{a} :

$$(0.26) \quad \mathbf{b} \cdot \mathbf{b} = -\rho^2 (a^t)^2 + P^2 \sum_{m=x,y,z} (a^m)^2 \leq 0$$

Because \mathbf{a} is causal, we have

$$(0.27) \quad (a^t)^2 \geq \sum_{m=x,y,z} (a^m)^2$$

The constraint on ρ and P in 0.26 comes in the case of equality in 0.27, in which case we have

$$(0.28) \quad -\rho^2 + P^2 \leq 0$$

and since $\rho > 0$ this amounts to

$$(0.29) \quad \rho \geq |P|$$

Finally, we can revisit the case of the stress-energy tensor that gives negative pressure, $T^{ij} = -\Lambda g^{ij}$, where Λ is a positive scalar. In this case, if we apply the DEC to some causal vector \mathbf{a} we get

$$(0.30) \quad b^i = -T^{ij} g_{jk} a^k$$

$$(0.31) \quad = \Lambda g^{ij} g_{jk} a^k$$

$$(0.32) \quad = \Lambda a^i$$

Since \mathbf{b} is a positive scalar multiplied by a causal vector, it too must be causal, so this stress-energy tensor satisfies the DEC.

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