EINSTEIN EQUATION: TRYING THE RICCI TENSOR AS A SOLUTION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 21; Box 21.1.

We can now start looking at a derivation of the Einstein equation, which is the generalization of Newton's formula for the gravitational force. In Newtonian theory, gravity is an attractive, conservative, inverse-square force so (apart from the sign) it is mathematically identical to the electrostatic force, which means we can write a differential form of Newton's gravitational theory using Gauss's law. That is

$$\nabla\cdot\mathbf{g}=-4\pi G\rho \tag{1}$$

where $\mathbf{g}$ is the gravitational field, ρ is the mass density and G is the gravitational constant. The minus sign occurs because gravity is attractive, whereas the electric force for like charges is repulsive. Because the force is conservative, $\mathbf{g}$ can be written as the gradient of a potential so an alternative form of the equation is

$$\nabla\cdot(-\nabla\Phi)=-\nabla^{2}\Phi=-4\pi G\rho \tag{2}$$

or

$$\boxed{\nabla^{2}\Phi=4\pi G\rho} \tag{3}$$

The derivation of the Einstein equation is, like so many derivations in relativity, based on a plausibility argument. We want to find a tensor equation that generalizes Newton's equation, and we want this tensor equation to reduce to Newton's equation in the weak field limit.

First, the generalization of Newtonian mass density ρ is the stress-energy tensor T^{ij} we can try replacing the RHS of 3 by κT^{ij} where κ is a scalar constant. Since the RHS is now a rank-2 tensor, the LHS must also be a rank-2 tensor, so we must have an equation like

$$G^{ij} = \kappa T^{ij} \tag{4}$$

where the form of G^{ij} needs to be determined. To do this, think about what we want the theory to do. The idea behind general relativity is that the energy density in a region of space should determine the curvature of the space in that region. The Riemann tensor and the metric tensor describe the curvature of space-time, so it makes sense that G^{ij} could depend on these two tensors.

Suppose we try to express G^{ij} solely in terms of the Riemann tensor. What other constraints can we impose to narrow things down? First, since the Riemann tensor is rank 4 and G^{ij} is rank 2, we'll need to contract the Riemann tensor to get rid of 2 of its indices. One candidate is the Ricci tensor, defined as the contraction of the Riemann tensor over its first and third indices:

$$R^a_{\ bac} = g^{ad} R_{dbac} \tag{5}$$
$$\equiv R_{bc} \tag{6}$$

To use R_{bc}, we need to raise both its indices, so we get

$$R^{ij} = g^{ib} g^{jc} R_{bc} \tag{7}$$
$$= g^{ib} g^{jc} R_{cb} \tag{8}$$
$$= g^{ic} g^{jb} R_{bc} \tag{9}$$
$$= R^{ji} \tag{10}$$

where in the third line, we've swapped the dummy indices b and c. Since T^{ij} is symmetric, G^{ij} must also be symmetric, but since $R^{ij} = R^{ji}$, this condition is satisfied.

From conservation of energy and momentum, we know that $\nabla_i T^{ij} = 0$, so we must also have $\nabla_i G^{ij} = 0$ (since κ is a constant). This is where we run into a snag. The condition $\nabla_i G^{ij} = 0$ must apply everywhere, in every reference frame, so it must apply the origin of a locally inertial frame (LIF). In a LIF, the Riemann tensor reduces to

$$R_{nj\ell m} = \frac{1}{2}\left(\partial_\ell \partial_j g_{mn} + \partial_m \partial_n g_{j\ell} - \partial_\ell \partial_n g_{jm} - \partial_m \partial_j g_{\ell n}\right) \tag{11}$$

The Ricci tensor is then

$$R^{ik} = g^{ij} g^{km} R_{jm} \tag{12}$$

$$= g^{ij} g^{km} g^{\ell n} R_{nj\ell m} \tag{13}$$

$$= \frac{1}{2} g^{ij} g^{km} g^{\ell n} \left(\partial_\ell \partial_j g_{mn} + \partial_m \partial_n g_{j\ell} - \partial_\ell \partial_n g_{jm} - \partial_m \partial_j g_{\ell n}\right) \tag{14}$$

In a LIF, the first derivatives of g^{ij} are all zero (by definition of the LIF), so

$$\nabla_i R^{ik} = \frac{1}{2} g^{ij} g^{km} g^{\ell n} \nabla_i \left(\partial_\ell \partial_j g_{mn} + \partial_m \partial_n g_{j\ell} - \partial_\ell \partial_n g_{jm} - \partial_m \partial_j g_{\ell n}\right) \tag{15}$$

In a LIF, the total derivative ∇_i reduces to the ordinary derivative ∂_i so we get

$$\nabla_i R^{ik} = \frac{1}{2} g^{ij} g^{km} g^{\ell n} \left(\partial_i \partial_\ell \partial_j g_{mn} + \partial_i \partial_m \partial_n g_{j\ell} - \partial_i \partial_\ell \partial_n g_{jm} - \partial_i \partial_m \partial_j g_{\ell n}\right) \tag{16}$$

The indexes in the first term can be relabelled by swapping i with ℓ and j with n to give

$$\frac{1}{2} g^{ij} g^{km} g^{\ell n} \partial_i \partial_\ell \partial_j g_{mn} = \frac{1}{2} g^{\ell n} g^{km} g^{ij} \partial_\ell \partial_i \partial_n g_{mj} \tag{17}$$

which is the negative of the third term (since $g_{mj} = g_{jm}$ and the order of the partial derivatives doesn't matter), so these two terms cancel and we're left with

$$\nabla_i R^{ik} = \frac{1}{2} g^{ij} g^{km} g^{\ell n} \left(\partial_i \partial_m \partial_n g_{j\ell} - \partial_i \partial_m \partial_j g_{\ell n}\right) \tag{18}$$

The only way this can be identically zero is if we could swap n with j in the first term and have it equal the negative of the second term. However, if we try this, we get

$$\frac{1}{2} g^{ij} g^{km} g^{\ell n} \partial_i \partial_m \partial_n g_{j\ell} = \frac{1}{2} g^{in} g^{km} g^{\ell j} \partial_i \partial_m \partial_j g_{n\ell} \tag{19}$$

The partial derivatives match up with those in the second term, but the product of the three metric tensors doesn't, so in general this isn't zero, meaning that

$$\nabla_i R^{ik} \neq 0 \tag{20}$$

Thus setting $G^{ij} = R^{ij}$ won't work, and we'll need to try something else.

PINGBACKS

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