EINSTEIN TENSOR AND EINSTEIN EQUATION

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 21; Box 21.2.

Our first attempt at constructing a tensor generalization of Newton's law of gravitation failed because the total derivative of the Ricci tensor is not, in general, zero. Recall that the general idea is to look for an equation of form

$$G^{ij}=\kappa T^{ij} \tag{1}$$

where G^{ij} depends on the Riemann tensor and the metric. At this stage, there's no particular reason to select any special form for this tensor, but if possible, we'd like it to be linear in the Riemann tensor. Also remember that G^{ij} must satisfy the conditions:

(1) It must be symmetric: ($G^{ij}=G^{ji}$).
(2) It must be of rank 2.
(3) Its total derivative must be zero: $\nabla_j G^{ij}=0$.

To that end, let's try a formula as follows:

$$R^{ij}+bg^{ij}R+\Lambda g^{ij} \tag{2}$$

where R is the curvature scalar

$$R=g^{ab}R_{ab} \tag{3}$$

and b and Λ are constants. Note that although R is a scalar, it is *not* a constant, so we can't just merge the last two terms.

This form for G^{ij} satisfies the first two conditions above, since everything on the RHS is of rank 2 and is also symmetric. So it just remains to show that the total derivative is zero. Since $\nabla_j g^{ij}=0$ always, the problem reduces to showing that

$$\nabla_j\left(R^{ij}+bg^{ij}R\right)=0 \tag{4}$$

To do this, we start with the Bianchi identity

$$\nabla_s R_{abmn}+\nabla_n R_{absm}+\nabla_m R_{abns}=0 \tag{5}$$

Multiplying through by $g^{gs}g^{am}g^{bn}$ we get (the metrics can be taken inside the derivatives since their derivatives are zero, so they act as constants):

$$\nabla_s g^{gs}g^{am}g^{bn}R_{abmn}+\nabla_n g^{gs}g^{am}g^{bn}R_{absm}+\nabla_m g^{gs}g^{am}g^{bn}R_{abns}=0 \quad (6)$$

In the first term, we have

$$\begin{aligned} g^{am}g^{bn}R_{abmn} &= g^{bn}R^m_{bmn} && (7)\\ &= g^{bn}R_{bn} && (8)\\ &= R && (9) \end{aligned}$$

so we get

$$\nabla_s g^{gs}R+\nabla_n g^{gs}g^{am}g^{bn}R_{absm}+\nabla_m g^{gs}g^{am}g^{bn}R_{abns}=0 \quad (10)$$

We can now use the symmetries of the Riemann tensor to simplify the last two terms. In the second term, we use $R_{absm}=R_{smab}=-R_{msab}$ and in the third term we use $R_{abns}=R_{nsab}=-R_{nsba}$:

$$\nabla_s g^{gs}R-\nabla_n g^{gs}g^{am}g^{bn}R_{msab}-\nabla_m g^{gs}g^{am}g^{bn}R_{nsba}=0 \quad (11)$$

The Ricci tensor with lowered indices is

$$R_{ab}=g^{cd}R_{dacb} \quad (12)$$

so to raise both its indices we have

$$R^{gs}=g^{ga}g^{sb}g^{mn}R_{namb} \quad (13)$$

Comparing this with the second term in 11 we can map the indices in that term as follows: $m\to n$, $s\to a$, $a\to m$, $n\to s$ and $b\to b$. Thus the second term is the same as

$$\begin{aligned} \nabla_n g^{gs}g^{am}g^{bn}R_{msab} &= \nabla_s g^{ga}g^{sb}g^{mn}R_{namb} && (14)\\ &= \nabla_s R^{gs} && (15) \end{aligned}$$

Similarly, in the third term we can map $n\to n$, $s\to a$, $b\to m$, $a\to b$ and $m\to s$ to get

$$\begin{aligned} \nabla_m g^{gs}g^{am}g^{bn}R_{nsba} &= \nabla_s g^{ga}g^{sb}g^{mn}R_{namb} && (16)\\ &= \nabla_s R^{gs} && (17) \end{aligned}$$

Thus 11 becomes

$$\nabla_s g^{gs} R - 2\nabla_s R^{gs} = 0 \tag{18}$$

or

$$\nabla_s \left(R^{gs} - \frac{1}{2} g^{gs} R \right) = 0 \tag{19}$$

Thus from 2 we can satisfy $\nabla_j \left(R^{ij} + b g^{ij} R + \Lambda g^{ij} \right) = 0$ if $b = -\frac{1}{2}$. In practice, G^{ij} is defined as just the first two terms:

$$\boxed{G^{ij} \equiv R^{ij} - \frac{1}{2} g^{ij} R} \tag{20}$$

This is known as the *Einstein tensor* and the equation

$$\boxed{G^{ij} + \Lambda g^{ij} = \kappa T^{ij}} \tag{21}$$

is the *Einstein equation*. This is the general relativistic replacement for Newton's law of gravity.

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