

## EINSTEIN EQUATION: ALTERNATIVE FORM

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 21; Box 21.3.

The general relativistic generalization of Newton's law of gravity is

$$G^{ij} + \Lambda g^{ij} = \kappa T^{ij} \quad (1)$$

where the Einstein tensor is defined in terms of the Ricci tensor and the curvature scalar as

$$G^{ij} \equiv R^{ij} - \frac{1}{2}g^{ij}R \quad (2)$$

We can write this in a different form that is sometimes easier to use in calculations. Eliminating  $G^{ij}$  we have

$$R^{ij} - \frac{1}{2}g^{ij}R + \Lambda g^{ij} = \kappa T^{ij} \quad (3)$$

Multiplying both sides by  $g_{ij}$  we get

$$g_{ij}R^{ij} - \frac{1}{2}g_{ij}g^{ij}R + \Lambda g_{ij}g^{ij} = \kappa g_{ij}T^{ij} \quad (4)$$

Because the tensor  $g_{ij}$  is the inverse of  $g^{ij}$ , their product gives the identity matrix of rank 4 (this can be seen by doing the calculation in a local inertial frame where  $g_{ij} = \eta_{ij}$  and noting that since it's a tensor equation, it's valid in all coordinate systems). That is

$$g_{ij}g^{jk} = \delta_i^k \quad (5)$$

so if we contract the  $\delta_j^k$  tensor we just sum up its diagonal elements and since these are all 1 (and there are four rows), we get

$$\delta_k^k = 4 \quad (6)$$

Returning to 4 we get

$$g_{ij}R^{ij} - 2R + 4\Lambda = \kappa g_{ij}T^{ij} \quad (7)$$

Since the curvature scalar is given by

$$R \equiv g_{ij}R^{ij} \quad (8)$$

and the stress-energy scalar is

$$T \equiv g_{ij}T^{ij} \quad (9)$$

we get

$$-R + 4\Lambda = \kappa T \quad (10)$$

Multiplying this by  $-\frac{1}{2}g^{ij}$  and subtracting from 3 we have

$$R^{ij} - \Lambda g^{ij} = \kappa \left( T^{ij} - \frac{1}{2}g^{ij}T \right) \quad (11)$$

Isolating the Ricci tensor gives us the alternative form of the Einstein equation:

$$\boxed{R^{ij} = \kappa \left( T^{ij} - \frac{1}{2}g^{ij}T \right) + \Lambda g^{ij}} \quad (12)$$

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