EINSTEIN EQUATION FOR A PERFECT FLUID

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 21; Problem 21.3.

The Einstein equation can be written as

$$R^{ij}=\kappa\left(T^{ij}-\frac{1}{2}g^{ij}T\right)+\Lambda g^{ij} \tag{1}$$

For a perfect fluid the stress-energy tensor is

$$T^{ij}=(\rho_0+P_0)\,u^iu^j+P_0g^{ij} \tag{2}$$

where u^i is the four-velocity, ρ_0 is the energy density and P_0 is the pressure. We have for the stress-energy scalar:

$$\begin{aligned}
T &= (\rho_0+P_0)\,g_{ij}u^iu^j+P_0g_{ij}g^{ij} && (3)\\
&= -(\rho_0+P_0)+4P_0 && (4)\\
&= 3P_0-\rho_0 && (5)
\end{aligned}$$

since $g_{ij}g^{ij}=4$ and $g_{ij}u^iu^j=-1$ in any coordinate system.

The stress energy term in 1 is therefore

$$\begin{aligned}
T^{ij}-\frac{1}{2}g^{ij}T &= (\rho_0+P_0)\,u^iu^j+P_0g^{ij}-\frac{1}{2}g^{ij}\,(3P_0-\rho_0) && (6)\\
&= (\rho_0+P_0)\,u^iu^j+\frac{1}{2}g^{ij}\,(\rho_0-P_0) && (7)
\end{aligned}$$

In a local orthogonal frame (LOF) in which the fluid is at rest $u^t=1$, $u^i=0$ for $i=x,y,z$ and $g^{ij}=\eta^{ij}$, so

$$\begin{aligned}
T^{tt}-\frac{1}{2}g^{tt}T &= \rho_0+P_0-\frac{1}{2}\left(\rho_0-P_0\right) && (8)\\
&= \frac{1}{2}\left(\rho_0+3P_0\right) && (9)\\
T^{xx}-\frac{1}{2}g^{xx}T &= \frac{1}{2}\left(\rho_0-P_0\right) && (10)\\
T^{yy}-\frac{1}{2}g^{yy}T &= \frac{1}{2}\left(\rho_0-P_0\right) && (11)\\
T^{zz}-\frac{1}{2}g^{zz}T &= \frac{1}{2}\left(\rho_0-P_0\right) && (12)
\end{aligned}$$

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