GRAVITY CAN'T EXIST IN 3 SPACETIME DIMENSIONS EITHER

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 21; Problem 21.6.

We've seen that the Einstein equation doesn't allow gravity to exist in 2 spacetime dimensions. Here we'll look at a demonstration that gravity also cannot exist in 3 spacetime dimensions of t, x and y.

Because of the symmetries of the Riemann tensor, there are six independent, (possibly) non-zero components in 3 dimensions. We can take these components to be R_{xtxt}, R_{ytyt}, R_{xtyt}, R_{txxy}, R_{txyt} and R_{tyxy}. As before, we use the Einstein equation

$$R^{ij}=\kappa\left(T^{ij}-\frac{1}{2}g^{ij}T\right)+\Lambda g^{ij} \tag{0.1}$$

with $\Lambda=0$ and consider a vacuum so that $T^{ij}=T=0$, meaning that $R^{ij}=0$. We'll look at a local inertial frame (LIF), where the metric is $g^{ij}=\eta^{ij}$. Then we get

$$\begin{aligned} R_{ij} &= R^{a}_{\ iaj} && (0.2)\\ &= \eta^{ak}R_{kiaj} && (0.3)\\ &= -R_{titj}+R_{xixj}+R_{yiyj} && (0.4)\end{aligned}$$

Now we look at the 6 independent components of R_{ij}. Because $R_{ijkm}=-R_{jikm}=-R_{ijmk}$, any component with either the first two indices or last two indices equal is zero, so we get

$$\begin{aligned} R_{tt} &= R_{xtxt}+R_{ytyt}=0 && (0.5)\\ R_{xx} &= -R_{txtx}+R_{yxyx}=0 && (0.6)\\ R_{yy} &= -R_{tyty}+R_{xyxy}=0 && (0.7)\\ R_{tx} &= R_{ytyx}=0 && (0.8)\\ R_{ty} &= R_{xtxy}=0 && (0.9)\\ R_{xy} &= -R_{txty}=0 && (0.10)\end{aligned}$$

The last 3 equations show that 3 of the Riemann components are zero. The first 3 equations can be rewritten using the symmetries of the Riemann tensor:

$$R_{tt} = R_{xtxt} + R_{ytyt} = 0 \tag{0.11}$$
$$R_{xx} = -R_{xtxt} + R_{xyxy} = 0 \tag{0.12}$$
$$R_{yy} = -R_{ytyt} + R_{xyxy} = 0 \tag{0.13}$$

Solving these equations gives

$$R_{xtxt} = R_{ytyt} = R_{xyxy} = 0 \tag{0.14}$$

Thus all 6 components of the Riemann tensor are zero, showing that 3d spacetime must be flat and gravity cannot exist in 3 spacetime dimensions. (As usual, a tensor equation valid in a LIF is valid in all coordinate systems, so the conclusion is general.)