EINSTEIN EQUATION FOR AN EXPONENTIAL METRIC

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 21, Problem 21.8.

Consider the metric:

$$ds^2 = -e^{2gx}dt^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

We'll have a look at what the Einstein equation has to say about gravity in a spacetime using this metric. First, we'll find the Christoffel symbols using the method of comparing the two forms of the geodesic equation. The geodesic equation is

$$\frac{d}{d\tau}\left(g_{aj}\frac{dx^j}{d\tau}\right) - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^a}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0 \tag{2}$$

The following equation is formally equivalent to this (where a dot above a symbol means the derivative with respect to τ):

$$\ddot{x}^m + \Gamma_{ij}^m \dot{x}^j \dot{x}^i = 0 \tag{3}$$

Since the metric is diagonal and none of its components depends on y or z, the LHS of 2 is identically zero for $a = y$ or $a = z$, so all Christoffel symbols with any index being y or z is zero.

Now look at $a = t$. We get from 2, since g_{ij} doesn't depend on t:

$$\frac{d}{d\tau}\left(g_{tj}\frac{dx^j}{d\tau}\right) = 0 \tag{4}$$

$$-2ge^{2gx}\dot{x}\dot{t} - e^{2gx}\ddot{t} = 0 \tag{5}$$

$$\ddot{t} + 2g\dot{x}\dot{t} = 0 \tag{6}$$

Comparing with 3 we see that

$$\Gamma^t_{\ tx} = \Gamma^t_{\ xt} = g \tag{7}$$

Now for $a = x$:

$$\ddot{x} - \frac{1}{2}\left(\partial_x g_{tt}\right)\dot{t}^2 = 0 \tag{8}$$

$$\ddot{x} + ge^{2gx}\dot{t}^2 = 0 \tag{9}$$

$$\Gamma^x_{tt} = ge^{2gx} \tag{10}$$

These are the only non-zero Christoffel symbols. We can get an expression for the acceleration of a particle in its rest frame by noting that

$$\ddot{x} = -ge^{2gx}\dot{t}^2 \tag{11}$$

The four velocity of a particle at rest (so $dx = dy = dz = 0$) must satisfy

$$u^i u_i = -1 \tag{12}$$

$$= g_{ij}u^i u^j \tag{13}$$

$$= g_{tt}\dot{t}^2 \tag{14}$$

$$= -e^{2gx}\dot{t}^2 \tag{15}$$

$$\dot{t}^2 = e^{-2gx} \tag{16}$$

Plugging into 11 we get

$$\ddot{x} = -g \tag{17}$$

That is, the acceleration in the x direction is a constant, so this would seem to indicate a uniform gravitational field. However, let's apply the Einstein equation and see what we get. We'll need the Riemann tensor in order to get the Ricci tensor, so we'll use the definition of the former:

$$R^i{}_{j\ell m} \equiv -\left[\partial_m \Gamma^i{}_{\ell j} - \partial_\ell \Gamma^i{}_{mj} + \Gamma^k{}_{\ell j}\Gamma^i{}_{km} - \Gamma^k{}_{mj}\Gamma^i{}_{\ell k}\right] \tag{18}$$

Since all Christoffel symbols with a y or z index are zero, the only possibly non-zero components of $R_{ij\ell m}$ are those containing only x or t, and due to the symmetry relations, the only independent component is

$$R_{txtx} = g_{ti}R^i{}_{xtx} \tag{19}$$

$$= e^{2gx}\left[\partial_x \Gamma^t{}_{tx} - \partial_t \Gamma^t{}_{xx} + \Gamma^k{}_{tx}\Gamma^t{}_{kx} - \Gamma^k{}_{xx}\Gamma^t{}_{tk}\right] \tag{20}$$

$$= e^{2gx}\left[0 - 0 + \Gamma^t{}_{tx}\Gamma^t{}_{tx} - 0\right] \tag{21}$$

$$= g^2 e^{2gx} \tag{22}$$

Now for the Ricci tensor. We have

$$R_{ab} = R^{i}{}_{aib} \tag{23}$$
$$= g^{ij}R_{jaib} \tag{24}$$

Since the metric is diagonal, the upstairs components are just the reciprocals of the downstairs ones, so

$$g^{tt} = -e^{-2gx} \tag{25}$$
$$g^{xx} = g^{yy} = g^{zz} = 1 \tag{26}$$

and we get

$$R_{tt} = g^{xx}R_{xtxt} = g^2e^{2gx} \tag{27}$$
$$R_{xx} = g^{tt}R_{txtx} = g^{tt}R_{xtxt} = -g^2 \tag{28}$$
$$R_{xt} = R_{tx} = g^{ab}R_{bxat} = 0 \tag{29}$$

where in the last line we see that R_{bxat} can be non-zero only if $b = t$ and $a = x$ but since g^{ab} is diagonal, $g^{xt} = 0$. All components of R_{ab} involving y or z indices are zero because all components of R_{abcd} involving y or z indices are zero. To use the Ricci tensor in the Einstein equation, we need the upstairs version, which is

$$R^{tt} = g^{ta}g^{tb}R_{ab} = e^{-4gx}\left(g^2e^{2gx}\right) = g^2e^{-2gx} \tag{30}$$
$$R^{xx} = g^{xa}g^{xb}R_{ab} = R_{xx} = -g^2 \tag{31}$$

In a vacuum (assuming $\Lambda = 0$) the stress-energy tensor $T^{ij} = 0$ and the Einstein equation says that

$$R^{ab} = 0 \tag{32}$$

The only way this can be true is if $g = 0$, meaning that there is no gravitational field.

More generally, suppose that there is some fluid in the region so that $T^{ij} \neq 0$. In that case

$$R^{tt} = g^2 e^{-2gx} = 8\pi G\left(T^{tt} + \frac{1}{2}e^{-2gx}T\right) \tag{33}$$

$$R^{xx} = -g^2 = 8\pi G\left(T^{xx} - \frac{1}{2}T\right) \tag{34}$$

$$R^{yy} = 0 = 8\pi G\left(T^{yy} - \frac{1}{2}T\right) \tag{35}$$

$$R^{zz} = 0 = 8\pi G\left(T^{zz} - \frac{1}{2}T\right) \tag{36}$$

where the stress-energy scalar is

$$T = g_{ij}T^{ij} \tag{37}$$

$$= -e^{2gx}T^{tt} + T^{xx} + T^{yy} + T^{zz} \tag{38}$$

From 35 and 36 we have

$$T^{yy} = T^{zz} = \frac{T}{2} \tag{39}$$

so from 38 we have

$$-e^{2gx}T^{tt} + T^{xx} = 0 \tag{40}$$

However, if we multiply 33 by e^{2gx} and add it to 34 we get

$$0 = 8\pi G\left(T^{tt}e^{2gx} + T^{xx}\right) \tag{41}$$

Combining the last two equations we get

$$T^{tt} = T^{xx} = 0 \tag{42}$$

Plugging 39 into 34 we get

$$T^{yy} = T^{zz} = \frac{g^2}{8\pi G} \tag{43}$$

This result agrees with the correction to Moore's problem 21.8 given in the errata.

In a perfect fluid at rest, T^{tt} is the energy density and the diagonal components T^{ii} for $i = x, y, z$ are the pressures in each of the 3 directions. If these results are to be believed, the fluid has zero energy ($T^{tt} = 0$) and no pressure in the x direction, but a non-zero pressure in the y and z directions. It's hard to imagine how a fluid could have zero energy and yet still exert pressure, so this would appear to be why the results are absurd, as stated by Moore.