# VACUUM STRESS-ENERGY AND THE COSMOLOGICAL CONSTANT

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Reference: Moore, Thomas A., *A General Relativity Workbook*, University Science Books (2013) - Chapter 21, Problem 21.9.

The Einstein equation is

$$R^{ij}=8\pi G\left(T^{ij}-\frac{1}{2}g^{ij}T\right)+\Lambda g^{ij} \tag{1}$$

Up to now, we've usually taken $\Lambda=0$, since we know from the Newtonian limit that $\Lambda$ must be very small. If $\Lambda\neq 0$, the Newtonian limit becomes

$$\nabla^{2}\Phi=4\pi G\rho-\Lambda \tag{2}$$

so $\Lambda$ acts as a negative mass density, that is, it adds a repulsive term into the gravitational force. Einstein originally introduced it to counter the attractive force of gravity on a cosmological scale, since at the time it was believed that the universe was static (neither expanding nor contracting) and if gravity were purely attractive, the universe would be contracting.

At the moment, the universe is believed to be expanding so $\Lambda$ is believed to be non-zero and positive, although still small enough that its effects are not noticeable on the scale of the solar system (or indeed on a galactic scale). Because of this, $\Lambda$ is called the *cosmological constant*.

We can include $\Lambda$ within the stress-energy tensor by defining a *vacuum stress-energy* as

$$T_{vac}^{ij}=-\frac{\Lambda}{8\pi G}g^{ij} \tag{3}$$

We can define a vacuum stress-energy scalar:

$$T_{vac}\equiv g_{ij}T_{vac}^{ij} \tag{4}$$

$$=-\frac{\Lambda}{8\pi G}g_{ij}g^{ij} \tag{5}$$

$$=-\frac{4\Lambda}{8\pi G} \tag{6}$$

Therefore

$$T^{ij}_{vac} - \frac{1}{2} g^{ij} T_{vac} = -\frac{\Lambda}{8\pi G} g^{ij} + \frac{2\Lambda}{8\pi G} g^{ij} \tag{7}$$
$$= \frac{\Lambda}{8\pi G} g^{ij} \tag{8}$$

and we can write 1 as

$$R^{ij} = 8\pi G \left( T^{ij} - \frac{1}{2} g^{ij} T + T^{ij}_{vac} - \frac{1}{2} g^{ij} T_{vac} \right) \tag{9}$$
$$\equiv 8\pi G \left( T^{ij}_{all} - \frac{1}{2} g^{ij} T_{all} \right) \tag{10}$$

where $T^{ij}_{all}$ includes the stress-energy from the mass-energy density and the vacuum.

The dominant energy condition is a constraint placed on the stress-energy tensor so that observers in any local orthogonal frame will measure the fluid's speed to be less than the speed of light. The condition is that if $a^i$ is any four-vector that is *causal*, that is, it satisfies the conditions

$$\mathbf{a} \cdot \mathbf{a} \leq 0 \tag{11}$$
$$a^t > 0 \tag{12}$$

then we require the stress-energy tensor $T^{ij}$ to satisfy the condition that if

$$b^i = -T^{ij} g_{jk} a^k \tag{13}$$

then $\mathbf{b}$ is also a causal four-vector. For the vacuum stress-energy this condition says

$$b^i = -T^{ij}_{vac} g_{jk} a^k \tag{14}$$
$$= \frac{\Lambda}{8\pi G} g^{ij} g_{jk} a^k \tag{15}$$
$$= \frac{\Lambda}{8\pi G} \delta^i_{\ k} a^k \tag{16}$$
$$= \frac{\Lambda}{8\pi G} a^i \tag{17}$$

That is, $b^i$ is just a positive (if $\Lambda > 0$) constant multiplied by $a^i$, so if $a^i$ is causal, then $b^i$ must also be causal. Thus $T^{ij}_{vac}$ satisfies the dominant energy condition.

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